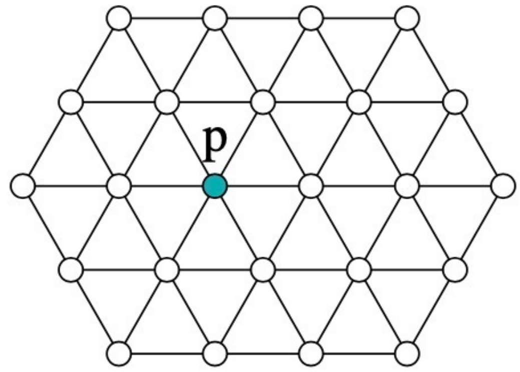
9.1. Levin-Gu model (2012)

2D SPT protected by $G = \mathbb{Z}_2$.

- Ising paramagnet.



$$H_0 = -\sum_p \sigma_p^x$$

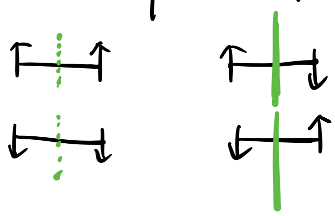
$$|\Psi_0\rangle = \bigotimes_p |\sigma_p^x = 1\rangle$$

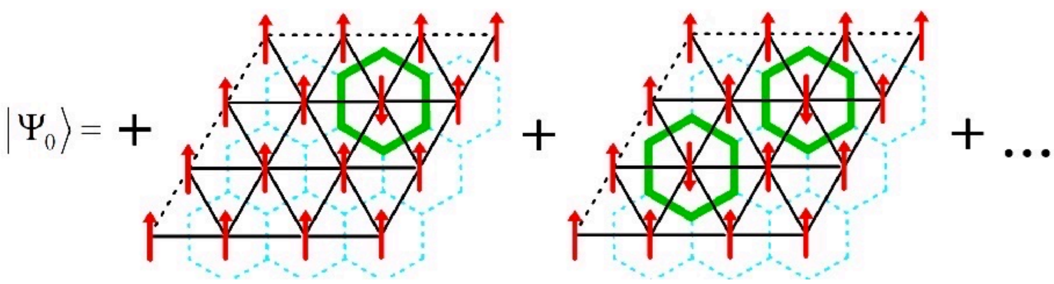
$$= \bigotimes_p \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_p + |\downarrow\rangle_p\right)$$

$$\propto \sum_{\{\sigma_p^z = \pm 1\}} |\{\sigma_p^z\}\rangle$$

$$\propto \sum_{\substack{\text{DW conf.} \\ (\text{even}, \text{even})}} |\text{DW conf.}\rangle$$

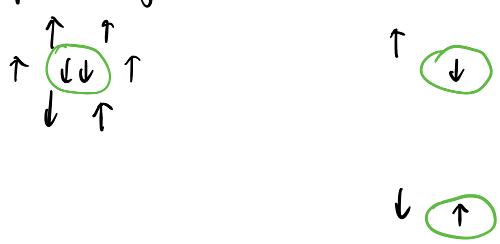
Domain wall picture :





On the plane:

spin conf. $\xrightarrow{2:1}$ DW conf.



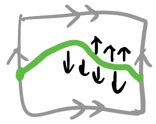
On the torus :

spin conf. $\xrightarrow{2:1}$ DW conf. (even, even)

(odd, even)

2:1

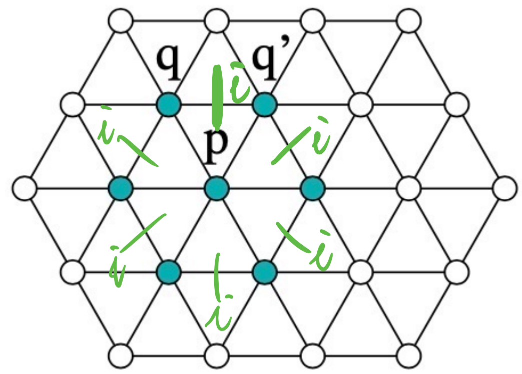
Spin conf. $\longrightarrow$ DW conf. (even, odd)
(odd, odd)

? $\longmapsto$ 

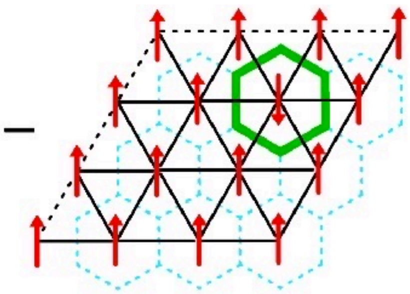
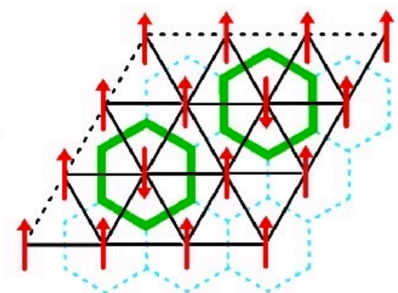
- Levin-Gu state.

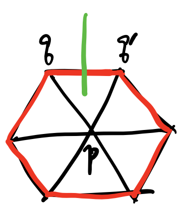
$$H_1 = -\sum_p B_p$$

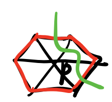
$$B_p = -\sigma_p^x \prod_{\langle pqq' \rangle} i^{\frac{1-\sigma_q^z \sigma_{q'}^z}{2}}$$

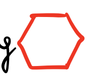


$$|\Psi_1\rangle = \sum_{\substack{\{\text{DW conf}\} \\ (\text{even, even})}} (-1)^{\#(\text{DW})} \, |\text{DW conf}\rangle$$

$|\Psi_1\rangle = -$  $+$  $+ \ldots$



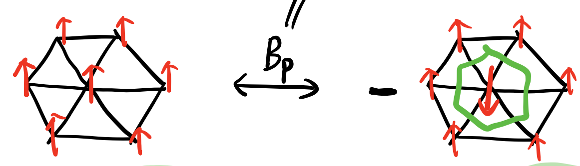
$$-\prod_{\langle pqq' \rangle} \underbrace{i^{\frac{1-\overset{=\pm 1}{\sigma_q^z \sigma_{q'}^z}}{2}}}_{\begin{cases} 1, & \text{if } \sigma_q^z \sigma_{q'}^z = +1 \Leftrightarrow \text{No DW for } \langle qq' \rangle \\ i, & \text{if } \sigma_q^z \sigma_{q'}^z = -1 \Leftrightarrow \text{DW for } \langle qq' \rangle \end{cases}} = -\, i^{\#(\text{DW}) \text{ crossing boundary of}}$$ 

$= \begin{cases} -\, i^2 = 1, & \text{if } \#(\text{DW}) \text{ crossing} \end{cases}$  $= 2 \mod 4$

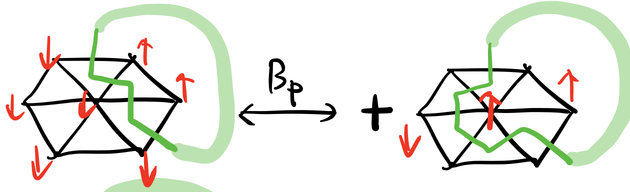
$\quad\;\; -\, i^0 = -1, \text{ if } \ldots \quad = 0 \mod 4$

$$= (-1)^{\text{changes of } \#(\text{total DW}) \text{ under } B_p}$$

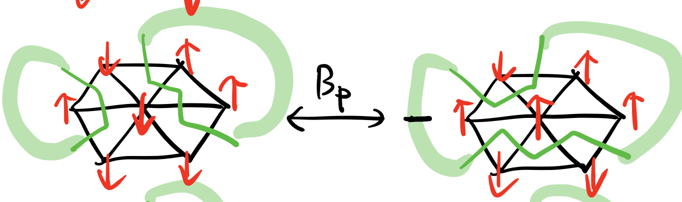
$(\pm) \cdot \sigma_p^x$

$\#(\mathrm{DW}) = 0$: 

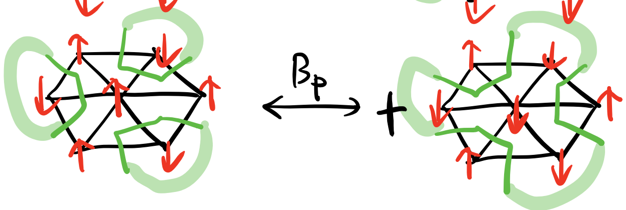
$\#(\text{total DW})$: $0 \to 1$

$\#(\mathrm{DW}) = 2$: 

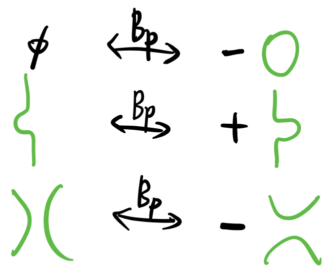
$1 \to 1$

$\#(\mathrm{DW}) = 4$: 

$2 \to 1$

$\#(\mathrm{DW}) = 6$: 

$3 \to 1$

In summary (in the continuum): 

$$|\Psi_1\rangle = \sum_{\{\sigma_p^z\}} (-1)^{\#(\mathrm{DW})} \left|\{\sigma_p^z\}\right\rangle$$

$$= \sum_{\substack{\text{DW conf}\\ c}} (-1)^{\#(\text{DW in } c)} \left| c \right\rangle$$

$$B_p \left| c \right\rangle = (-1)^{\#(\text{DW in } c+\partial p) - \#(\text{DW in } c)} \left| \underset{(\text{mod } 2)}{c+\partial p} \right\rangle$$

$$\Rightarrow B_p \, (-1)^{\#(\text{DW in } c)} \left| c \right\rangle = (-1)^{\#(\text{DW in } c+\partial p)} \left| c+\partial p \right\rangle$$

$$\Rightarrow B_p \sum_c (-1)^{\#(\text{DW in } c)} \left| c \right\rangle = \sum_c (-1)^{\#(\text{DW in } c+\partial p)} \left| c+\partial p \right\rangle$$

$$= \sum_c (-1)^{\#(\text{DW in } c)} \left| c \right\rangle$$

$$\Rightarrow B_p \left| \Psi_1 \right\rangle = \left| \Psi_1 \right\rangle$$

$\Rightarrow$ $|\Psi_1\rangle$ is the GS of H_1.

$$\begin{cases} |\Psi_1\rangle = \sum_{\{\sigma_p^z\}} (-1)^{\#DW} |\{\sigma_p^z\}\rangle \\ H_1 = -\sum_p B_p \end{cases}$$

nonlocal sign.

difference of nonlocal signs → local

local term

9.2. Gauging a global symmetry.

System with onsite global symmetry G:

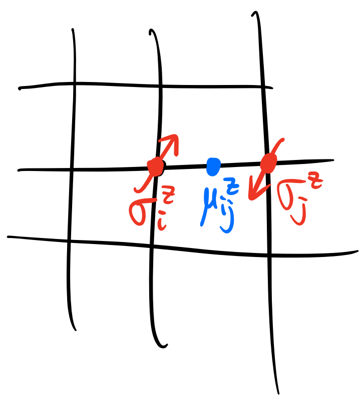
$$U(g) = \bigotimes_{\text{site } i} U_i(g) \quad \text{acting on} \quad \mathcal{H} = \bigotimes_i \mathcal{H}_i$$

$$[U(g), H] = 0, \quad \forall g \in G.$$

↓ gauge

G gauge theory with local gauge symmetry / redundancy.

Gauge $\underline{\text{Ising paramagnet}}$ to $\underline{\text{toric code}}$ on square lattice.
(global $\mathbb{Z}_2$ symm) (${\mathbb{Z}_2}$ gauge symm)



$$H_0 = -\sum_i \sigma_i^x, \quad |\Psi_0\rangle = \bigotimes_i |\sigma_i^x = 1\rangle$$

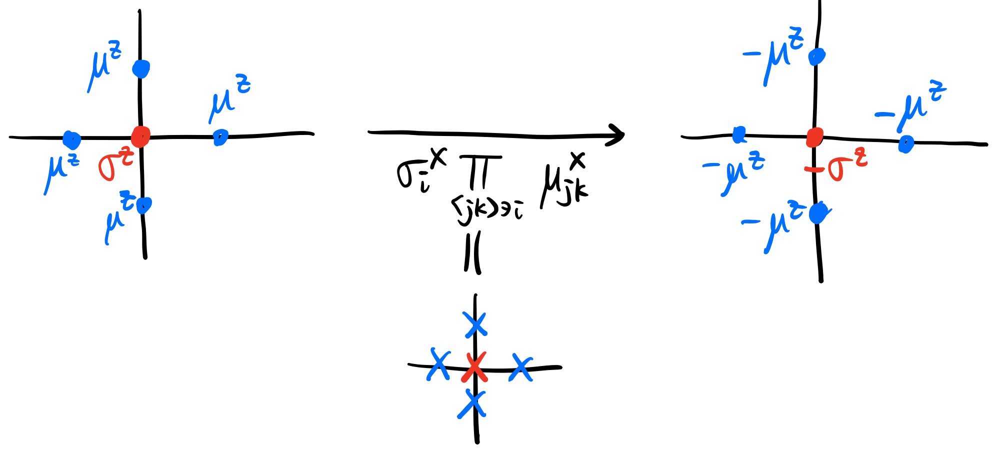
$$U = \bigotimes_i \sigma_i^x$$

$$[U, H_0] = 0$$

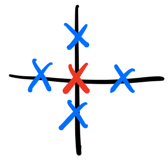
gauging procedure:

① Adding $\underset{U(1)}{\mathbb{Z}_2}$ gauge field $\underset{}{\mu_{ij}^z}$ (A_{ij}) on link $\langle ij \rangle$

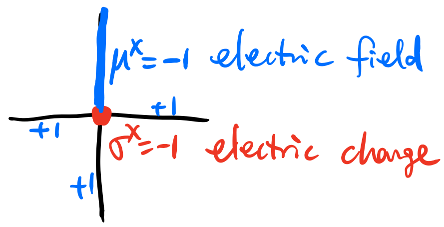
② Enforcing local gauge symmetry (Gauss law) on the total Hilbert space.



local gauge transformation

Gauss law:  $= \sigma_i^x \prod_{\langle jk\rangle \ni i} \mu_{jk}^x = 1$

$$\nabla \cdot \vec{E} = \rho$$

eg: 

$$\underbrace{\left(\sigma_i^x \prod_{\langle jk\rangle \ni i} \mu_{jk}^x\right)}_{\text{local gauge symm transf. at site } i} |\Psi_{phy}\rangle = |\Psi_{phy}\rangle$$

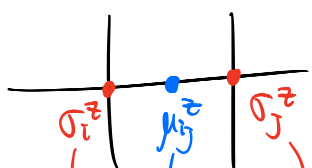
$$\prod_i \underbrace{\left(\sigma_i^x \prod_{\langle jk\rangle \ni i} \mu_{jk}^x\right)}_{\substack{\text{local symm} \\ (\mathbb{Z}_2)^N}} = \prod_i \sigma_i^x = \underbrace{U}_{\substack{\text{global symm} \\ \mathbb{Z}_2}}$$

infinite # of local symm/redundancies!

③ Minimal coupling

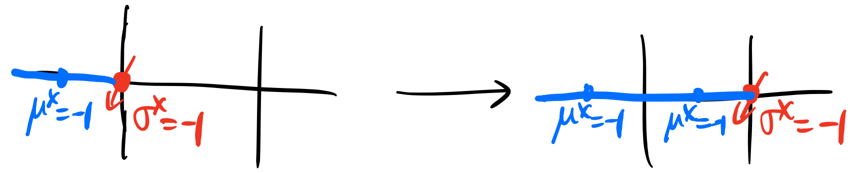
$$\sigma_i^z \sigma_j^z \longrightarrow \sigma_i^z \mu_{ij}^z \sigma_j^z$$

$$c_i^\dagger c_j \qquad\qquad c_i^\dagger e^{i A_{ij}} c_j$$



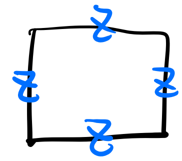
flip electric field

flip $\mathbb{Z}_2$ charge

eg. 

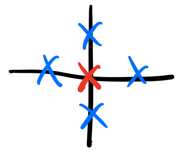
④ Adding zero flux condition and Gauss law to Hamiltonian.

$$-\sum_p \prod_{\langle ij\rangle\in\partial p} \mu_{ij}^z$$



$$\left(\oint_\Box dx^\mu A_\mu\right)^2 = \vec{B}^2$$

$$-\sum_i \sigma_i^x \prod_{\langle jk\rangle\ni i} \mu_{jk}^x$$



For Ising paramagnet:

$$H_0 = -\sum_i \sigma_i^x \quad \left(-J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z\right)$$

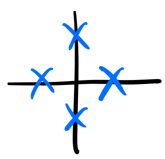
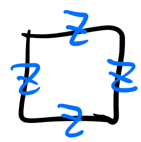
$\downarrow$ gauging

$$\tilde{H}_0 = -\sum_i \sigma_i^x \quad \left(-J\sum_{\langle ij\rangle}\sigma_i^z\mu_{ij}^z\sigma_j^z\right)$$

$$-\sum_i \sigma_i^x \prod_{\langle jk\rangle\ni i}\mu_{jk}^x - \sum_p \prod_{\langle ij\rangle\in\partial p}\mu_{ij}^z$$

$\sigma_i^x=+1$ to minimize E

$$= -\sum_i \prod_{\langle jk\rangle\ni i}\mu_{jk}^x - \sum_p \prod_{\langle ij\rangle\in\partial p}\mu_{ij}^z$$

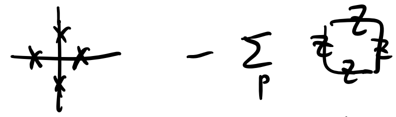
$$= -\sum_i$$  $$-\sum_p$$ 

= toric code model.

Excitations: $\quad H_0 = -\sum_i \sigma_i^x, \quad U = \prod_i \sigma_i^x$

$$\begin{cases} GS: & |\sigma_i^x=+1\rangle \\ \text{excited state}: & |\sigma_i^x=-1\rangle \text{ for some } i \end{cases}$$

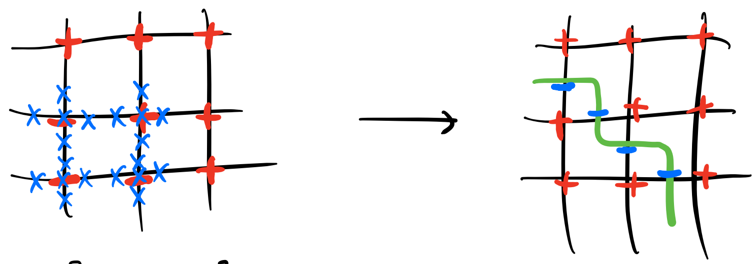
↳ $\mathbb{Z}_2$ charge

$\tilde{H}_0 = -\sum_i$  $-\sum_p$

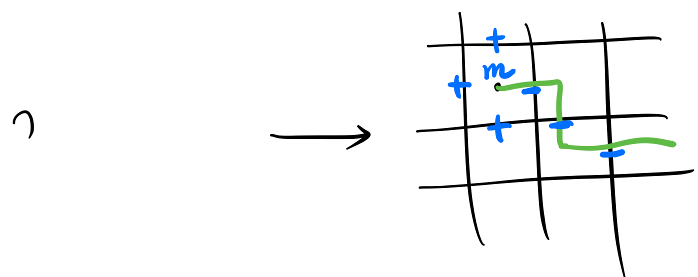
↳ $\mathbb{Z}_2$ charge e ↳ $\mathbb{Z}_2$ flux m (new excitations after gauging)

New excitations $m \longleftrightarrow$ endpoints of domain walls.

Consider conf. $\{\sigma_i^z = \pm 1, \mu_{ij}^z = 1\}$, we can do gauge transformation $\sigma_i^x \prod_{ij} \sigma_{ij}^x$ for i with $\sigma_i^z = -1$:



$\sigma_i^z = \pm 1, \mu_{ij}^z = 1$ $\sigma_i^z = +1, \mu_{ij}^z = \pm 1$



DW = ∂(region under symm action)

$\Rightarrow$ DW is closed loop

open DW,

endpoints = m excitations.

Gauging:
- expand Hilbert space
- promote global symm to local gauge symm/redundancy.
- introduce new excitations = gauge flux

9.3. Gauging Levin-Gu model to double semion model

$\parallel$

twisted ~~quantum~~ double model with $G=\mathbb{Z}_2$

Q: How to show that Levin-Gu state is different from Ising paramagnet?

A: Gauge global $\mathbb{Z}_2$ symmetry:

Ising paramagnet $\longrightarrow$ toric code $= D(\mathbb{Z}_2)$ $\quad \{1, e, m, f\}$

Levin-Gu $\longrightarrow$ double semion $= D^{\omega_3}(\mathbb{Z}_2)$ $\quad \{1, e, s, \bar{s}\}$

$$H_0 = -\sum_p \sigma_p^x$$

$\downarrow$ gauging

$$\widetilde{H}_0 = -\sum_p \sigma_p^x O_p - \sum_{\langle pqr\rangle} \mu_{pq}^z \mu_{qr}^z \mu_{rq}^z, \qquad O_p = \prod_{\langle pqr\rangle} \frac{1+\mu_{pq}^z \mu_{qr}^z \mu_{rq}^z}{2}$$

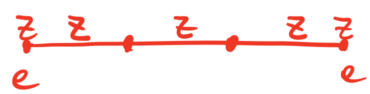
$$H_1 = -\sum_p B_p, \qquad B_p = -\sigma_p^x \prod_{\langle pqq'\rangle} i^{\frac{1-\sigma_q^z \sigma_{q'}^z}{2}}$$

$\downarrow$ gauging

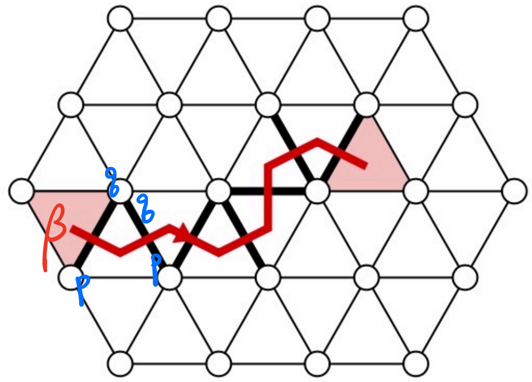
$$\widetilde{H}_1 = -\sum_p \widetilde{B}_p O_p - \sum_{\langle pqr\rangle} \mu_{pq}^z \mu_{qr}^z \mu_{rq}^z,$$

$$\widetilde{B}_p = -\sigma_p^x \prod_{\langle pqq'\rangle} i^{\frac{1-\sigma_q^z \mu_{qq'}^z \sigma_{q'}^z}{2}}$$

$$\begin{cases} \text{charge excitation } e: & \sigma_p^x = -1 \text{ for } \widetilde{H}_0 \\ & B_p = -1 \text{ for } \widetilde{H}_1 \\ \text{flux excitation } m: & \mu_{pq}^z \mu_{qr}^z \mu_{rq}^z = -1 \end{cases}$$

Ribbon operator for e: 

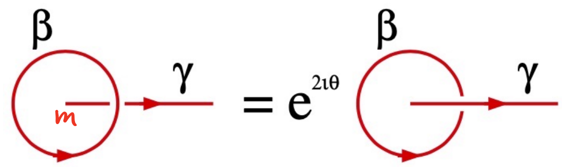
Flux excitations m:

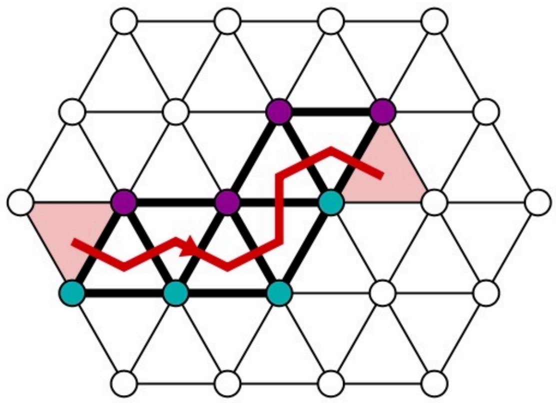


$$V_\beta^0 = \prod_{\langle pq\rangle \perp \beta} \mu^x_{pq}$$

$$V^0_\beta V^0_\gamma = V^0_\gamma V^0_\beta$$

$\Rightarrow$ m is a boson

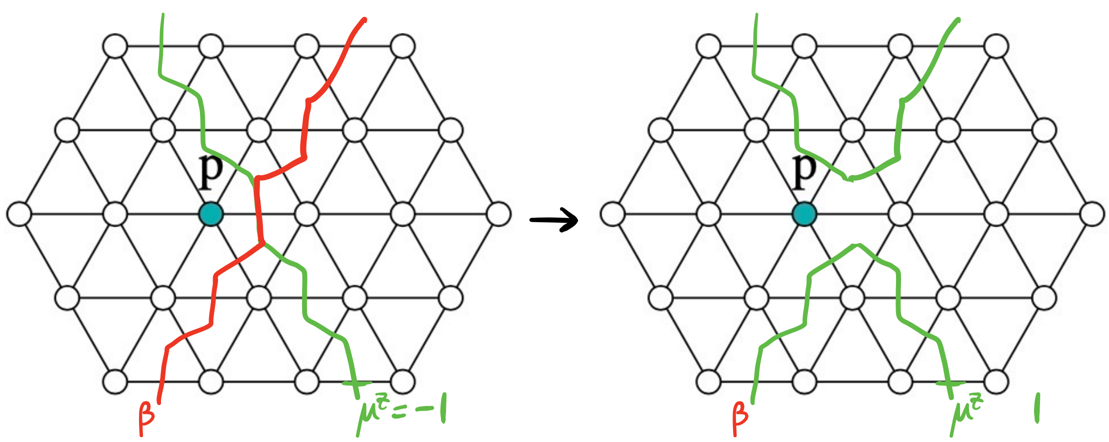




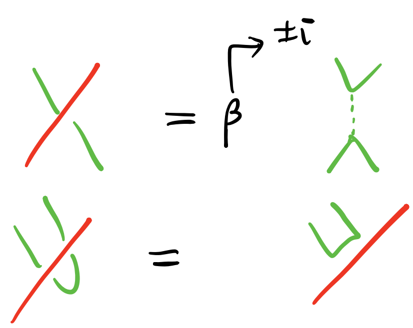
$$V_\beta^1 = \prod_{\langle pq\rangle \perp \beta} \mu^x_{pq} \cdot \prod_{\langle pqq'\rangle, r} i^{\frac{1-\sigma^z_q \mu^z_{qq'} \sigma^z_{q'}}{2}} \quad ($$

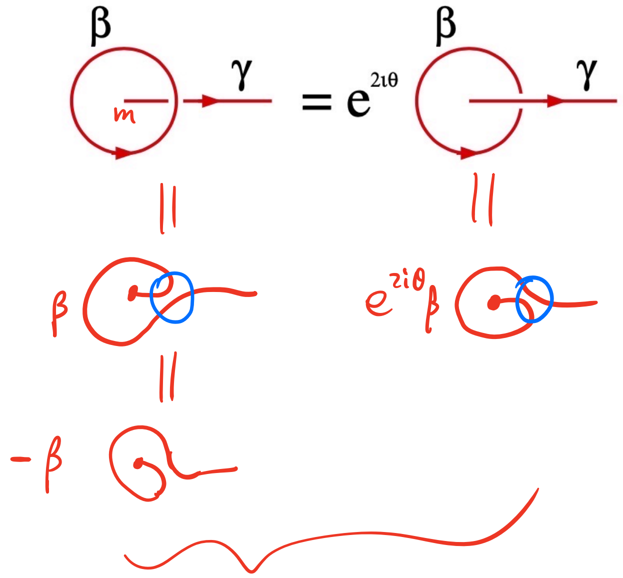
$$\cdot \prod_{\langle pqq'\rangle, l} (-1)^{\widetilde{s}_{pqq'}} \cdot \prod_{\langle pqq'\rangle \in \beta} (1 + \mu^z_{pq}\mu^z_{qq'}\mu^z_{pq'})/2$$

$$\widetilde{s}_{pqq'} = \frac{1}{4}(1 - \sigma^z_p \mu^z_{pq} \sigma^z_q)(1 + \sigma^z_p \mu^z_{pq'} \sigma^z_{q'})$$

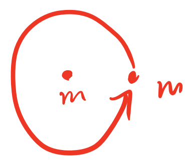


$$V_\beta^1 \, |\mu^z = \pm 1\rangle = \beta \, |\mu'^z = \pm 1\rangle$$





$\Rightarrow e^{2i\theta} = -1$

$\Rightarrow$ 

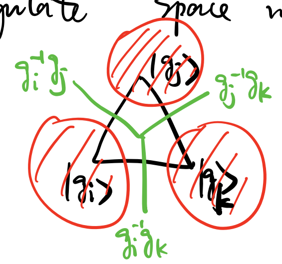
$= -1$

$\Rightarrow$ m is semion

9.4. Group cohomology model for G-SPT.

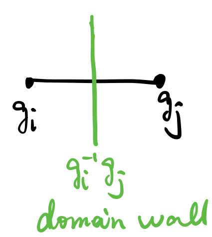
(1) General SPT wavefunction

Triangulate space manifold.

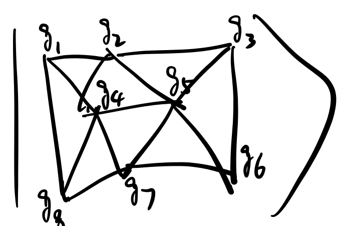


$g_i, g_j, g_k \in G.$

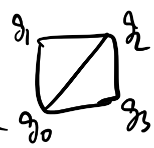
$U(g)\,|g_i\rangle = |g\,g_i\rangle$

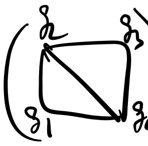


$$|\Psi\rangle = \sum_{\{g_i\}} \Psi(c)\;|c\rangle$$

$|c\rangle =$ 

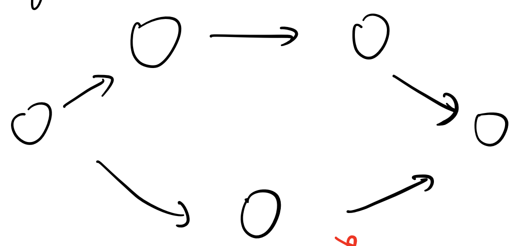
Retriangulations (Pachner move): change the shapes of DWs.

$$\Psi\Big($$ 

$$\Big) = \nu_3(g_0, g_1, g_2, g_3)\ \Psi\Big($$ 

$$\Big)$$

Pentagon eq:



$$(d\nu_3)(g_0, g_1, \cdots, g_4) = \frac{\nu_3(g_1, g_2, g_3, g_4)\ \nu_3(g_0, g_1, g_3, g_4)\ \nu_3(g_0, g_1, g_2, g_3)}{\nu_3(g_0, g_2, g_3, g_4)\ \nu_3(g_0, g_1, g_2, g_4)}$$

Symmetry condition:

$$U(g)\,|\Psi\rangle = |\Psi\rangle$$

$$\Rightarrow U(g) \sum_{\{g_i\}} \Psi(\{g_i\})\,|\{g_i\}\rangle = \sum_{\{g_i\}} \Psi(\{g_i\})\,|\{g_i\}\rangle$$

$$\Rightarrow \Psi(\{g g_i\}) = \Psi(\{g_i\})$$

$$\Rightarrow \nu_3(g g_0, \cdots, g g_3) = \nu_3(g_0, \cdots, g_3)$$

$$\nu_3 \in Z^3(G, U(1))$$

Symmetric local unitary transf:

$$|\{g_i\}\rangle \longrightarrow \prod_{\Delta} \nu_2(g_0, g_1, g_2)^{\pm 1}\,|\{g_i\}\rangle$$

$$\nu_3(g_0, \cdots, g_3) \to \nu_3(g_0, \cdots, g_3) \cdot d\nu_2(g_0, \cdots, g_3)$$

$$(d\nu_2)(g_0, \cdots, g_3) = \frac{\nu_2(123)\ \nu_2(013)}{\nu_2(023)\ \nu_2(012)}$$

$$\nu_3 \sim \nu_3 \cdot d\nu_2$$

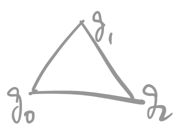
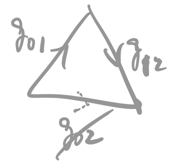
$$\nu_3 \in H^3(G, U(1)) = Z^3(G, U(1)) / B^3(G, U(1))$$

(2) Dijkgraaf-Witten = twisted gauge theory = gauged SPT.

2+1 D: $$Z[M_3] = \sum_{\{g_i\}} \prod_{\Delta_3 \in M_3} \nu_3(\Delta_3)^{s(\Delta_3)}$$

$$\big\downarrow \text{gauging}$$

$$\tilde{Z}[M_3] = \sum_{\substack{\{g_{ij}\} \\ dg=1}} \prod_{\Delta_3 \in M_3} \nu_3(\Delta_3)^{s(\Delta_3)}$$

	SPT	DW with fixed background gauge field	DW
dof.	site $g_i \in G$ trivial bundle	link $g_{ij} \in G$ non-trivial bundle	link $g_{ij} \in G$ Σ nontrivial bundles
cocycles	homogeneous → $\nu_d(gg_0,\cdots,gg_d) = \nu_d(g_0,\cdots,g_d)$ $\nu_d(g_0,g_1,\cdots,g_d)$  $\nu_2(g_0,g_1,g_2)$		inhomogeneous $\nu_d(g_{01},g_{12},\cdots,g_{d-1d})$  $\nu_2(g_{01},g_{12})$
Z	$\frac{1}{\lvert G\rvert^N}\sum_{\{g_i\}}\prod_{\Delta}\nu_d(\Delta)^{s(\Delta)}$	$\prod_{\Delta}\nu_d(\Delta)^{s(\Delta)}$ (g_{ij} is fixed)	$\frac{1}{\lvert G\rvert^N}\sum_{\substack{\{g_{ij}\}\\ dg=1}}\prod_{\Delta}\nu_d(\Delta)^{s(\Delta)}$
$Z(M_{d-1}\times S^1)$	1	$\in U(1)$	$GSD(M_{d-1})$