8. Introduction to symmetry-protected topological (SPT) phases

	short-range entangled (SRE)	long-range entangled (LRE)
without symmetry	invertible TO	intrinsic TO (MTC)
with symmetry	SPT	SET (G+MTC) ↑ enriched

Note: Sometimes, are called invertible phases, in the sense that there exist inverse for these phase.

$$\begin{cases} \text{Topological phases} \rightarrow \text{Abelian monoid under stacking} \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \exists \text{ inverse} \\ \text{invertible phases} \rightarrow \text{Abelian group under stacking.} \end{cases}$$

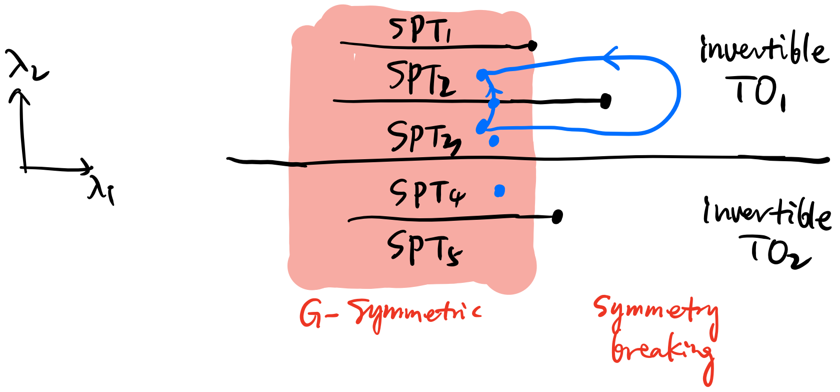
invertible phase
- invertible TO : Gapped states without anyons, but still can NOT be deformed into trivial product state.
- G-SPT : Gapped states without anyons, but still can NOT be deformed into trivial product state while preserving the symmetry G.

Known classification for invertible TO:

dim	0	1	2	3
bosonic	0	0	$\mathbb{Z}$ ↳ E_8 state	0
fermionic	$\mathbb{Z}_2$ ↓ bosonic/fermionic	$\mathbb{Z}_2$ ↓ Majorana chain	$\mathbb{Z}$ ↓ p_x+ip_y SC	0

Note: fermionic iTO = fermion SPT protected by $G_f = \mathbb{Z}_2^f = \{1, (-1)^F\}$

Schematic phase diagram :



Depending on microscopic dof & (non) interacting :

SPT	noninteracting	interacting
bosonic	/	bSPT → this chapter
fermionic	TI/TSC ↓ last chapter	fSPT → last chapter ?

Similar to TI/TSC , nontrivial properties of SPT is on the edge.

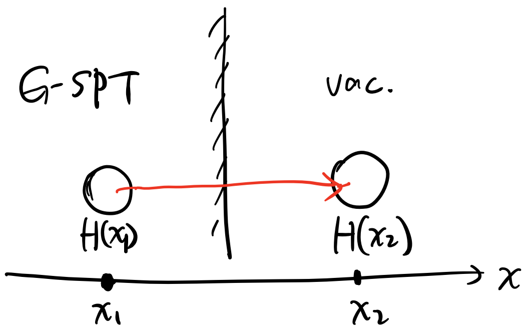
Possible edge states for SPT :

(1) G symmetry breaking

(2) anomalous SET (with edge GSD)

(3) gapless

~~(4)~~ unique gapped symmetric edge state .

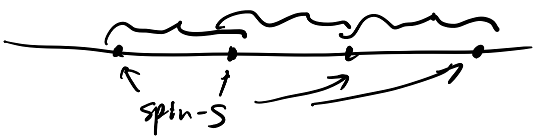


homotopy path $H(x)$: $H(x_1) \to H(x_2)$

8.1. Haldane chain (1983)

Consider 1D antiferromagnetic spin-S chain:

$$\hat{H} = \sum_i \vec{\hat{S}_i} \cdot \vec{\hat{S}}_{i+1}$$



classical configuration:

$$\vec{S}_i \approx (-1)^i S \, \hat{n}_i + \vec{l}_i$$

$\uparrow$ low E mode $\quad$ $\uparrow$ high E mode

$\int D\vec{l}_i \rightarrow$ involves $\hat{n}_i$

$$Z = \int \left(\prod_i D\vec{n}_i\right) e^{-S[\vec{n}_i]}$$

$$S = \int d^2x \, \frac{1}{2g^2} (\partial_\mu \hat{n})^2 + i\theta W[\vec{n}] \quad \begin{cases} \theta = 2\pi S \\ W = \int d^2x \, \frac{1}{4\pi} \hat{n} \cdot (\partial_t \hat{n} \times \partial_x \hat{n}) \end{cases}$$

$\uparrow$ NLσM $\quad$ $\uparrow$ θ term

Haldane conjecture:

$$H \text{ is } \begin{cases} \text{gapped} \\ \text{gapless} \end{cases} \text{ if } S \in \begin{cases} \mathbb{Z} \\ \mathbb{Z} + \frac{1}{2} \end{cases}$$

$\begin{cases} \text{If } S \in \mathbb{Z}, \quad Z = \int D\hat{n} \, e^{-S_{NL\sigma M}} \longrightarrow \text{gapped} \\ \text{If } S \in \mathbb{Z} + \frac{1}{2}, \end{cases}$

① $S = \frac{1}{2}$, Bethe ansatz

② $S \in \mathbb{Z} + \frac{1}{2}$, Lieb-Shultz-Mattis thm

③ $\int D\hat{n} \, (-1)^{W[\vec{n}]} e^{-S_{NL\sigma M}}$

Derivation of θ term.

① Path int for a single spin

$$\hat{\vec{S}} = \frac{1}{2} \hat{\vec{\sigma}}$$

Consider $\hat{H} = -\vec{n} \cdot \vec{S} = -\frac{1}{2} \vec{n} \cdot \hat{\vec{\sigma}}$

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$E = \pm\frac{1}{2}$, ground state is

$$|\vec{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \Rightarrow \langle\vec{n}|\hat{\vec{\sigma}}_i|\vec{n}\rangle = n_i$$

For a time evolution $|\vec{n}(t)\rangle$, the Berry phase

$$\gamma = -i\int_0^T dt \, \langle\vec{n}(t)| \frac{d}{dt} |\vec{n}(t)\rangle$$

$$= -i\int_0^T dt \left(\cos\frac{\theta(t)}{2}, e^{-i\phi(t)}\sin\frac{\theta(t)}{2}\right) \begin{pmatrix} \frac{d}{dt}\cos\frac{\theta(t)}{2} \\ \frac{d}{dt}\left[e^{i\phi(t)}\sin\frac{\theta(t)}{2}\right] \end{pmatrix}$$

$$= -i\int_0^T dt \left[-\frac{\theta'}{2}\sin\cancel{\frac{\theta}{2}\cos\frac{\theta}{2}} + i\phi'\sin^2\frac{\theta(t)}{2} + \cancel{\frac{\theta'}{2}\sin\frac{\theta}{2}}\cos\frac{\theta}{2}\right]$$

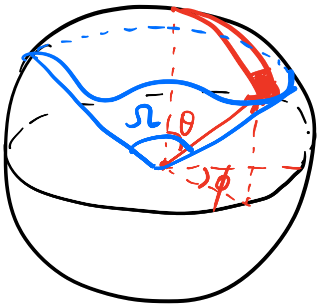
$$= \int_0^T dt \, \frac{1-\cos\theta(t)}{2} \, \frac{d\phi(t)}{dt}$$

$$= \frac{1}{2}\int d\phi \, [1-\cos\theta]$$

$$= \frac{1}{2}\int \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{2}\Omega[\vec{n}(t)]$$

↳ solid angle of trajectory of $\vec{n}(t)$ on S^2.



Another SU(2) invariant form of Ω is

$$\Omega = \int_0^1 d\rho \int_0^T dt \; \vec{n}\cdot(\partial_t\vec{n} \times \partial_\rho\vec{n})$$

with $\vec{n}(t,\rho) = \begin{cases} (0,0,1), & \text{if } \rho=0 \\ \vec{n}(t), & \text{if } \rho=1 \end{cases}$

WZ term for one spin

For general spin S:

$$\gamma = S\cdot\Omega[\vec{n}(t)]$$

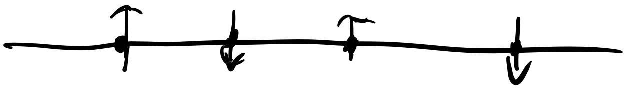
$$\gamma' = S\cdot(-\Omega') = -S(4\pi-\Omega) \underset{\substack{\uparrow \\ S\in\frac{1}{2}\mathbb{Z}}}{\overset{\text{mod } 2\pi}{=\!=}} S\Omega = \gamma$$

single spin :

$$Z = \int D\vec{n}(t)\, e^{iS\,\Omega[\vec{n}(t)]} + \cdots$$

$\uparrow$ Wess-Zumino term for (0+1)D

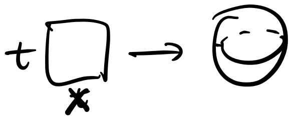
② AF spin chain :



$$Z = \int \left(\prod_i D\vec{n}_i(t)\right) e^{iS\sum_i \Omega[\vec{n}_i(t)]} + \cdots$$

$\vec{n}_i(t) = (-1)^i \vec{m}_i(t)$, s.t. $\vec{m}(x,t) := \vec{m}_i(t)$ is smooth

$$\begin{aligned} S\sum_i \Omega[\vec{n}_i(t)] &= S\sum_i \Omega[(-1)^i m_i(t)] \\ &= S\sum_i (-1)^i\, \Omega[m_i(t)] \\ &= S\sum_k \left(\Omega[m_{2k}(t)] - \Omega[m_{2k-1}(t)]\right) \\ &= \frac{S}{2}\int_0^L dx\, \frac{\partial}{\partial x}\Omega[m(x,t)] \\ &= \frac{S}{2}\int_0^L dx\, \frac{\delta\Omega}{\delta\vec{m}}\cdot\frac{\partial\vec{m}}{\partial x} \\ &= \frac{S}{2}\int dt\, dx\, (\vec{m}\times\partial_t\vec{m})\cdot\partial_x\vec{m} \\ &= 2\pi S\, \underbrace{\frac{1}{4\pi}\int dt\, dx\, \vec{m}\cdot(\partial_t\vec{m}\times\partial_x\vec{m})}_{\in W \text{ winding number } (T^2\to S^2)} \\ &= 2\pi W \end{aligned}$$



$$Z = \int D\vec{n}(x,t)\, e^{i2\pi S\, W[\vec{n}(x,t)]} + \cdots$$

$\downarrow$ θ term for (1+1)D.

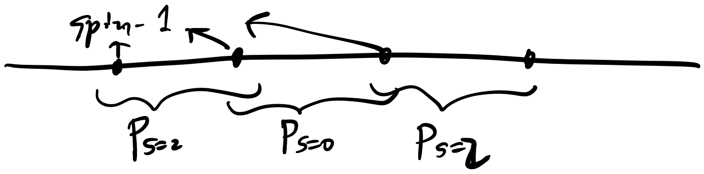
8.2. AKLT model

||

Affleck, Lieb, Kennedy, Tasaki (1987)

- Exactly solvable model for Haldane phase.

$$H_{AKLT} = \sum_i P_{S=2}(\vec{S}_i + \vec{S}_{i+1})$$



spin : $1 \otimes 1 = 0 \oplus 1 \oplus 2$ $\qquad \vec{S}^2 = S(S+1)$

$$(\vec{S}_i + \vec{S}_{i+1})^2 = \vec{S}_i^{\,2} + \vec{S}_{i+1}^{\,2} + 2\vec{S}_i \cdot \vec{S}_{i+1}$$

$$= 1\times(1+1) + 1\times(1+1) + 2\vec{S}_i \cdot \vec{S}_{i+1}$$

$$= 4 + 2\vec{S}_i \cdot \vec{S}_{i+1}$$

$$(\vec{S}_i + \vec{S}_{i+1})^2 = S_{tot}(S_{tot}+1) = \begin{cases} 0, & S_{tot}=0 \\ 2, & S_{tot}=1 \\ 6, & S_{tot}=2 \end{cases}$$

$$P_{S=2} = \frac{1}{24}(\vec{S}_i + \vec{S}_{i+1})^2 \cdot \left[(\vec{S}_i + \vec{S}_{i+1})^2 - 2\right] = \begin{cases} 0 & , \text{if } S_{tot}=0 \\ 0 & , \text{if } S_{tot}=1 \\ 1 & , \text{if } S_{tot}=2 \end{cases}$$

$$= \frac{1}{24}(4 + 2\vec{S}_i \cdot \vec{S}_{i+1})(4 + 2\vec{S}_i \vec{S}_{i+1} - 2)$$

$$= \underbrace{\frac{1}{2}\vec{S}_i \cdot \vec{S}_j}_{\text{Heisenberg}} + \underbrace{\frac{1}{6}(\vec{S}_i \cdot \vec{S}_j)^2 + \frac{1}{3}}_{\text{new terms}}$$

$$H = \sum_{\langle ij \rangle} \left[\cos\theta \, \vec{S}_i \cdot \vec{S}_j + \sin\theta \, (\vec{S}_i \cdot \vec{S}_j)^2\right]$$

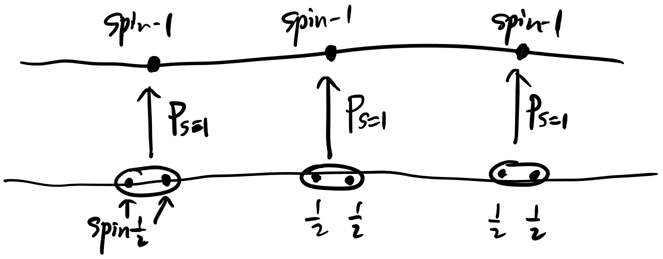
- AKLT state.

$H = \sum_{\langle ij \rangle} P_{S=2}(\vec{S}_i + \vec{S}_j)$ consists of projectors.

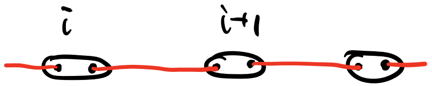
But $\left[P_{S=2}(\vec{S}_{i-1} + \vec{S}_i) \, , \, P_{S=2}(\vec{S}_i + \vec{S}_{i+1})\right] \neq 0$

There is a state $|AKLT\rangle$, s.t. $P_{S=2}(\vec{S}_i + \vec{S}_{i+1})|AKLT\rangle = 0, \forall i$.

Spin-$\frac{1}{2}$ representation :



$|AKLT\rangle =$



$$\underbrace{\tfrac{1}{2}\otimes(\tfrac{1}{2} \otimes \tfrac{1}{2})\otimes\tfrac{1}{2}}_{\tfrac{1}{2}\otimes 0\otimes\tfrac{1}{2}}$$

$$= \tfrac{1}{2}\otimes\tfrac{1}{2}$$

$$= 0\oplus 1$$

where

• = spin $\frac{1}{2}$

 $= \left(\frac{1}{2}\otimes\frac{1}{2} \xrightarrow{P_{s=1}} \text{spin } 1\right)$

 $= \left(\frac{1}{2}\otimes\frac{1}{2} \xrightarrow{P_{s=0}} \text{spin } 0\right)$

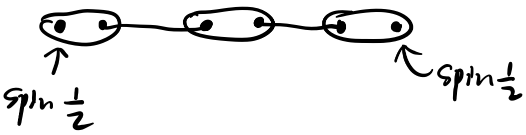
$\Rightarrow$ $P_{s=2}(\vec{S}_i+\vec{S}_{i+1})\,|AKLT\rangle = 0$.

$\Rightarrow$ $|AKLT\rangle$ is a ground state of H_{AKLT}.

Properties: (1) H_{AKLT} is gapped with GS $|AKLT\rangle$ (Haldane conjecture)

(2) H_{AKLT} and $|AKLT\rangle$ preserve $SO(3)_s$, time reversal, lattice inversion symmetries.

(3) $|AKLT\rangle$ has spin-$\frac{1}{2}$ edge state for open boundary condition.



GSD = 4

AKLT is an 1D SPT protected by $SO(3)_s$, time reversal, or lattice inversion symmetries.

- For general symm group G, we can construct AKLT-like state, s.t. the boundary is nontrivial preserving G.

projective rep :

$$U(g)\, U(h) = w_2(g,h)\, U(gh) , \quad w_2(g,h) \in U(1)$$

$$(g\cdot h)\cdot k = g\cdot(hk)$$

$$\Rightarrow \left[U(g)\cdot U(h)\right]\cdot U(k) = U(g)\cdot\left[U(h)\cdot U(k)\right]$$

$$\Rightarrow w_2(g,h)\, U(gh)\, U(k) = U(g)\cdot w_2(h,k)\, U(hk)$$

$$\Rightarrow w_2(g,h)\, w_2(gh,k)\, \underbrace{U(ghk)} = w_2(h,k)\, w_2(g,hk)\, \underbrace{U(ghk)}$$

$$\Rightarrow (d_2 w_2)(g,h,k) = \frac{w_2(h,k)\; w_2(g,hk)}{w_2(gh,k)\; w_2(g,h)} = 1$$

w_2 is called 2-cocycle of G.

If we do basis transf. $U'(g) := \mu_1(g)\, U(g)$, for $\mu_1 : G \to U(1)$.

$$\left.\begin{aligned} U'(g)\, U'(h) &= w_2'(g,h)\, U'(gh) \\ U(g)\, U(h) &= w_2(g,h)\, U(gh) \end{aligned}\right\}$$

$$\Rightarrow w_2'(g,h) = w_2(g,h) \cdot \underbrace{\frac{\mu_1(h)\, \mu_1(g)}{\mu_1(gh)}}_{=: (d_1\mu_1)(g,h)}$$

$$\Rightarrow w_2' = w_2 \cdot (d_1 \mu_1)$$

$\Rightarrow$ w_2 and w_2' give equivalent proj. rep.

Def. $H^2(G,U(1)) = \dfrac{Z^2(G,U(1))}{B^2(G,U(1))}$ 2^{nd} group cohomology of G, where

$Z^2(G,U(1)) := \{ w_2 : G^{\times 2} \to U(1) \mid d_2 w_2 = 1 \}$ → 2 cocycle

$B^2(G,U(1)) := \{ d_1 \mu_1 \mid \mu_1 : G \to U(1) \}$ → 2 coboundary

$H^2(G, U(1))$ is an Abelian group :

$$\left.\begin{array}{l} U(g)\,U(h) = \omega_2(g,h)\,U(gh) \\ U'(g) \quad \cdots \end{array}\right\}$$

$$\Rightarrow [U(g)\otimes U'(g)][U(h)\otimes U'(h)] = \underbrace{\omega_2(g,h)\,\omega_2'(g,h)}\,[U(gh)\otimes U'(gh)]$$

Nontrivial proj. rep must be higher dim.

$$\left.\begin{array}{l} U(g)\,U(h) = \omega_2(g,h)\,U(gh) \\ U(g) \in U(1) \end{array}\right\} \Rightarrow \omega_2(g,h) = \frac{U(g)\,U(h)}{U(gh)} \in B^2(G, U(1))$$

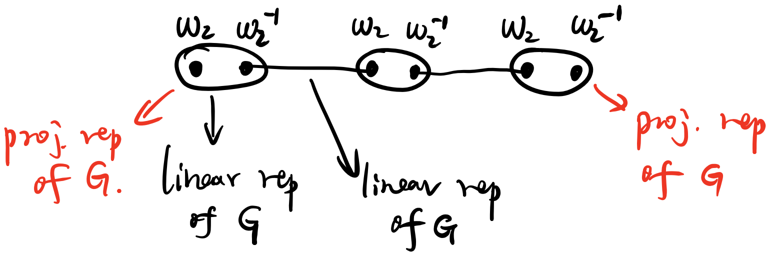
$$\Rightarrow [\omega_2] = [0].$$

Spin $\frac{1}{2}$: $\quad SO(3)$: $\omega_2(\pi,\pi) = -1$

$\mathbb{Z}_2^T$: $\omega_2(T,T) = -1$

projective reps.

AKLT-like state for G:



1D G-SPT is classified (at least) by $H^2(G, U(1))$.

$\cdots$ (at most) $\cdots$

$\Rightarrow$ 1D G-SPT is classified by $H^2(G, U(1))$.