7. Symmetries and classifications of free-fermion TI / TSC

7.1. Symmetries in quantum systems

Goal: Difference and relation of

① symmetries of operators acting on Hilbert space

② symmetries of physical states.

- Hilbert space $\neq$ physical states space

$\mathcal{H} \cong \mathbb{C}^N$, two states $|\psi\rangle$ and $|\psi'\rangle$ describes the same physical states iff $|\psi\rangle = z|\psi'\rangle$ for $0 \neq z \in \mathbb{C}$.

physical states space $P\mathcal{H} = \mathbb{C}P^{N-1} \cong (\mathcal{H} - \{0\})/\mathbb{C}^\times$.

<u>Ex</u>. spin $\frac{1}{2}$, $\mathcal{H} = \mathbb{C}^2 = \{a|\uparrow\rangle + b|\downarrow\rangle \mid a, b \in \mathbb{C}\} = \{\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2\}$

$\forall \begin{pmatrix} a \\ b \end{pmatrix}$:

① $a \neq 0$: $\begin{pmatrix} a \\ b \end{pmatrix} \sim \frac{1}{\sqrt{|a|^2+|b|^2}} \begin{pmatrix} |a| \\ |a| \cdot b/a \end{pmatrix} \sim \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix}$

② $a = 0$: $\begin{pmatrix} 0 \\ b \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} \sim \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix} \text{ with } \begin{pmatrix} 0 \leq \theta \leq \pi \\ 0 \leq \phi < 2\pi \end{pmatrix}$$

$\Rightarrow P\mathcal{H} \cong \mathbb{C}P^1 \cong S^2 \to$ Bloch sphere.

$$\dim_{\mathbb{R}} \mathcal{H} = 4 \underbrace{\quad , \quad}_{\substack{\text{normalization } -1 \\ U(1) \text{ phase } -1}} \dim_{\mathbb{R}} P\mathcal{H} = 2$$

- Symmetries of Hilbert space $\neq$ Symmetries of physical states.

$\mathrm{Aut}_{qtm}(P\mathcal{H})$: set of automorphisms of quantum systems.

transformation should preserve probability

$$|\langle\psi|\varphi\rangle|^2 = |\langle U\psi|U\varphi\rangle|^2$$

$\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$: set of unitary and antiunitary transformations on $\mathcal{H}$.

$$U: |\psi\rangle \mapsto U|\psi\rangle$$

U is $\mathbb{R}$-linear

$$\begin{cases} \text{unitary} : & U\big(a|\psi\rangle + b|\varphi\rangle\big) = a\,U|\psi\rangle + b\,U|\varphi\rangle \\ \text{anti-unitary} : & U\big(a|\psi\rangle + b|\varphi\rangle\big) = a^*\,U|\psi\rangle + b^*\,U|\varphi\rangle \end{cases}$$

Ex. spin $\frac{1}{2}$. $P\mathcal{H} = S^2$, $\mathcal{H} = \mathbb{C}^2$

$$\mathrm{Aut}_{qtm}(P\mathcal{H}) = \mathrm{Aut}_{qtm}(S^2) \cong O(3) \cong \mathbb{Z}_2 \times SO(3)$$

$$\mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) = \mathrm{Aut}_{\mathbb{R}}(\mathbb{C}^2) = \{\text{unitary/antiunitary transf. actioning on } \mathbb{C}^2\}$$
$$= U(2) \oplus U(2) = \mathbb{Z}_2 \times U(2)$$

Relation :

$$0 \to U(1) \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \xrightarrow{\pi} \mathrm{Aut}_{qtm}(P\mathcal{H}) \to 0$$

Wigner thm : Every quantum automorphism in $\mathrm{Aut}_{qtm}(P\mathcal{H})$ is induced by a unitary or anti-unitary operator in $\mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$ on Hilbert space $\mathcal{H}$.

$$1 \to U(\mathcal{H}) \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \xrightarrow{\phi} \mathbb{Z}_2 \longrightarrow 0$$

$$\phi(s) = \begin{cases} +1, & \text{if } s \text{ is unitary} \\ -1, & \text{-- -- antiunitary.} \end{cases}$$

- Physical symm and twisted symm.

$\mathrm{Aut}_{qtm}(P\mathcal{H})$: all symmetries of a q system.

Add Hamiltonian $\hat{H}$: smaller symmetry G.

$$\rho : G \to \mathrm{Aut}_{qtm}(P\mathcal{H})$$

twisted extension G^{tw} of G :

$$\begin{tikzcd} 1 \arrow[r] & U(1) \arrow[r] \arrow[d, equal] & G^{tw} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & U(1) \arrow[r] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r] & \mathrm{Aut}_{qtm}(P\mathcal{H}) \arrow[r] & 1 \end{tikzcd}$$

$\begin{cases} G: & \text{physical symmetry acting on physical states} \\ G^{tw}: & \text{virtual symmetry acting on Hilbert space.} \end{cases}$

Ex. $G = \mathbb{Z}_2^T = \{1, T\}$ is time reversal symmetry.

$$\begin{cases} \phi(1) = 1 \\ \phi(T) = -1 \end{cases}$$

$$1 \to U(1) \to G^{tw} \to \mathbb{Z}_2^T \to 1$$

G^{tw} is classified by $H^2(\mathbb{Z}_2^T, U(1)) = \mathbb{Z}_2 \ni \omega_2$

$$\begin{cases} (1) \; G^{tw} \cong \{zT \mid zT = Tz^{-1},\ z \in U(1),\ T^2 = 1\} \cong U(1) \rtimes \mathbb{Z}_2^T \\ (2) \; G^{tw} \cong \{zT \mid zT = Tz^{-1},\ z \in U(1),\ T^2 = -1\} \cong U(1) \rtimes_{\omega_2} \mathbb{Z}_2^T \end{cases}$$

7.2. 10-fold way of TI/TSC.

- unitary symmetries are NOT important in the classification of TI/TSC.

$$\hat{H} = \sum_{A,B} \hat{\psi}_A^\dagger \mathcal{H}_{A,B} \hat{\psi}_B \qquad \mathcal{H}^T \sim \mathcal{H}^*$$

$$\{\psi_A, \psi_B^\dagger\} = \delta_{AB}, \quad \{\psi_A, \psi_B\} = \{\psi_A^\dagger, \psi_B^\dagger\} = 0.$$

If we have a unitary symmetry:

$$\begin{cases} \hat{U} \hat{\psi}_A \hat{U}^\dagger = \sum_B U_{A,B}^\dagger \hat{\psi}_B \\ \hat{U} \psi_A^\dagger \hat{U}^\dagger = \sum_B \psi_B^\dagger U_{B,A} \end{cases} \qquad U \text{ is unitary matrix.}$$

$$\hat{U} \hat{H} \hat{U}^{-1} = \hat{H} \iff U \mathcal{H} U^\dagger = \mathcal{H}$$

We can block-diagonalize $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_\lambda \mathcal{H}_\lambda$$

For each λ, we have several copies of a fixed irrep. of G.

$$V_\lambda = V_\lambda^{(H)} \otimes V_\lambda^{(G)}$$

Hamiltonian acting on

Symmetry acting on

Within $V_\lambda^{(H)}$, there is No constraints for $\mathcal{H}$.

Ex. $G = SO(3)$. $V = \mathbb{C}^6 = \underset{\lambda=\text{spin-}1}{\overset{SO(3) \circlearrowright}{\mathbb{C}^3_\lambda}} \oplus \underset{\lambda=\text{spin-}1}{\overset{SO(3) \circlearrowright}{\mathbb{C}^3_\lambda}} = \mathbb{C}^2 \otimes \mathbb{C}^3$

$$[\mathcal{H}, SO(3)] = 0 \Rightarrow \left.\begin{array}{l} \mathcal{H} = \mathcal{H}_{2\times2} \otimes I_{3\times3} \\ U(g) = I_{2\times2} \otimes U_\lambda(g) \end{array}\right\}$$

- Antiunitary symmetries and 10-fold way.

(1) time reversal

$$\begin{cases} \hat{\mathcal{T}} \hat{\psi}_A \hat{\mathcal{T}}^{-1} = \sum_B (U_T^\dagger)_{A,B} \hat{\psi}_B \\ \hat{\mathcal{T}} \hat{\psi}_A^\dagger \hat{\mathcal{T}}^{-1} = \sum_B \hat{\psi}_B^\dagger (U_T)_{B,A} \\ \hat{\mathcal{T}} i \hat{\mathcal{T}}^{-1} = -i \end{cases} \qquad \mathcal{T} = U_T \cdot K$$

$$\hat{\mathcal{T}} \hat{H} \hat{\mathcal{T}}^{-1} = \hat{H} \Longleftrightarrow \boxed{U_T \mathcal{H}^* U_T^\dagger = \mathcal{H}}$$

$$\hat{\mathcal{T}}^2 = \pm 1 \Leftrightarrow U_T U_T^* = \pm 1$$

$$T = \begin{cases} 0, & \text{if no } \mathcal{T} \text{ symm.} \\ +1, & \text{if } \mathcal{T}^2 = +1 \\ -1, & \text{if } \mathcal{T}^2 = -1 \end{cases} \longrightarrow 3$$

(2) charge conjugation (particle-hole) symmetry.

$$\begin{cases} \hat{C} \hat{\psi}_A \hat{C}^{-1} = \sum_B (U_C^{*\dagger})_{AB} \hat{\psi}_B^\dagger \\ \hat{C} \hat{\psi}_A^\dagger \hat{C}^{-1} = \sum_B \hat{\psi}_B (U_C^*)_{BA} \\ \hat{C} i \hat{C}^{-1} = i \end{cases}$$

Comes from T (transpose)

$$\hat{C} \hat{H} \hat{C}^{-1} = \hat{H} \Longleftrightarrow \boxed{U_c \mathcal{H}^* U_c^\dagger = -\mathcal{H}}$$

$$C = \begin{cases} 0, & \text{if no } C \text{ symm.} \\ +1, & \text{if } C^2 = +1 \\ -1, & \text{if } C^2 = -1 \end{cases} \longrightarrow 3$$

(3) chiral (sublattice) symmetry.

$$\hat{S} = \hat{\mathcal{T}} \cdot \hat{C}$$

$$S = U_S = U_T \cdot U_C^*$$

$$\hat{S}\hat{H}\hat{S}^{-1} = \hat{H} \Leftrightarrow \boxed{U_S \mathcal{H} U_S^{+} = -\mathcal{H}}$$

$$S = \begin{cases} 0, & \text{if no } S \text{ symm.} \\ 1, & \text{if } S \text{ symm.} \end{cases}$$

if $T = C = 0$, $\quad S = T \cdot C = \begin{cases} 0 \\ 1 \end{cases}$

$T = 0, \pm \quad C = 0, \pm \quad \begin{cases} T = C = 0 \\ S = 1 \end{cases}$

In total, $3 \times 3 + 1 = 10$ classes

TABLE - "Ten Fold Way" [`CARTAN Classes'] Examples

Name (Cartan)	T	C	S= T C	Time evolution operator $U(t) = \exp\{itH\}$	Anderson Localization NLSM Manifold G/H [compact (fermionic) sector]	SU(2) spin con-served	Some Examples of Systems
A (unitary)	0	0	0	U(N)	U(2n)/U(n)xU(n)	yes/ no	IQHE Anderson
AI (orthogonal)	+1	0	0	U(N)/O(N)	Sp(4n) /Sp(2n)xSp(2n)	yes	Anderson
AII (symplectic)	-1	0	0	U(2N)/Sp(2N)	SO(2n)/SO(n)xSO(n)	no	*Quantum spin Hall Z2-Top.Ins. Anderson(spinorbit)*
AIII (chiral unitary)	0	0	1	U(N+M)/U(N)xU(M)	U(n)	yes/ no	Random Flux Gade SC
BDI (chiral orth.)	+1	+1	1	SO(N+M)/SO(N)xSO(M)	U(2n)/Sp(2n)	yes/ no	Bipartite Hopping Gade
CII (chiral sympl.)	-1	-1	1	Sp(2N+2M) /Sp(2N)xSp(2M)	U(n)/O(n)	no	Bipartite Hopping Gade
D	0	+1	0	O(N)	O(2n)/U(n)	no	(px+ipy)-wave 2D SC w/spin-orbit TQHE
C	0	-1	0	Sp(2N)	Sp(2n)/U(n)	yes	Singlet SC +mag.field (d+id)-wave SQHE
DIII	-1	+1	1	O(2N)/U(N)	O(n)	no	SC w/ spin-orbit He-3 B
CI	+1	-1	1	Sp(2N)/U(N)	Sp(2n)	yes	Singlet SC

(Ludwig 2015)

7.3. Examples of TI/TSC classification

2 complex classes:

A	:	TCS = 0 0 0
AIII	:	TCS = 0 0 1

8 real classes:

AI	:	TCS = + 0 0	insulators $G_f = U(1)_f$
AII	:	− 0 0	
D	:	0 + 0	Superconductors $G_f = \mathbb{Z}_2^f$
BDI	:	+ + 1	
DIII	:	− + 1	
C	:	0 − 0	superconductors with $SU(2)_s$ symm.
CI	:	+ − 1	
CII	:	− − 1	

- classification of 2D class A TI.

$$TCS = 000 \quad \swarrow \; \searrow \quad \mathcal{T}\text{ broken} \quad \text{insulator} \Longrightarrow \text{Chern insulator} \quad \mathbb{Z}$$

$$\hat{H} = \sum_{A,B} \hat{\psi}_A^\dagger H_{AB} \hat{\psi}_B$$

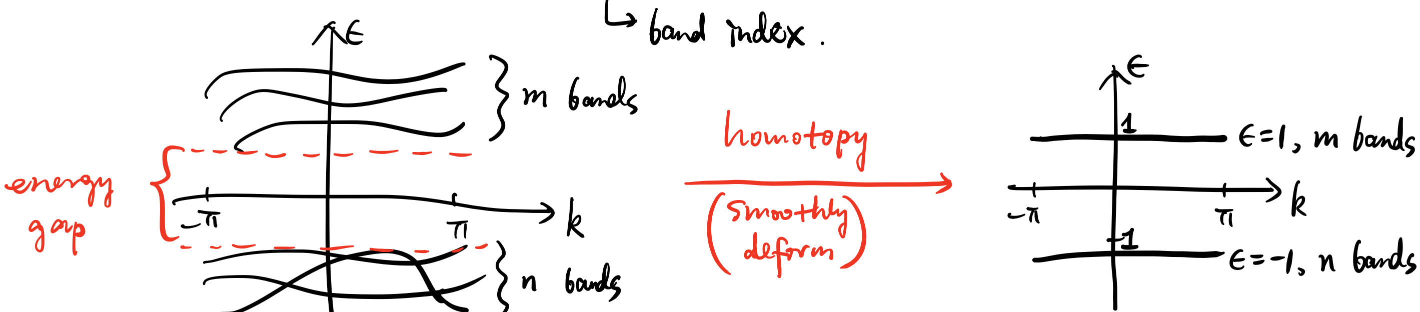
H_{AB} ↓ hermitian matrix $H^\dagger = H \Leftrightarrow H \in \mathfrak{u}(N) \Leftrightarrow e^{iH} \in U(N)$

$\{\text{class A Hamiltonians}\} = \mathfrak{u}(N)$

If we have translational symmetry,

$$\hat{H} = \sum_{\vec{k} \in BZ = T^2} \sum_{i=1}^{m+n} \epsilon_{ik} \psi_{ik}^\dagger \psi_{ik}$$

i ↳ band index.



$$H(k) = U(k)\begin{pmatrix} \epsilon_1(k) & & \\ & \ddots & \\ & & \epsilon_{m+n}(k)\end{pmatrix} U(k)^\dagger \longrightarrow \tilde{H}(k) = U(k)\begin{pmatrix} I_m & \\ & -I_n \end{pmatrix} U^\dagger(k)$$

wavefuntion unchanged
(U same)

$H \in u(m+n)$ simplified $\tilde{H} \in \dfrac{U(m+n)}{U(m)\times U(n)}$

For $U = \begin{pmatrix} U_m & \\ & U_n \end{pmatrix}$ with $U_m \in U(m)$, $U_n \in U(n)$,
$\tilde{H} = U \begin{pmatrix} I_m & \\ & -I_n \end{pmatrix} U^\dagger = \begin{pmatrix} U_m U_m^\dagger & \\ & -U_n U_n^\dagger \end{pmatrix} = \begin{pmatrix} I_m & \\ & -I_n \end{pmatrix}$

$$\{\text{class A simplified Hamiltonians}\} = \frac{U(m+n)}{U(m)\times U(n)} = C_0$$

2D class A TI

$\Leftrightarrow \quad \tilde{H}: \; BZ = T^2 \longrightarrow C_0$
$\qquad\qquad \vec{k} \longmapsto \tilde{H}(\vec{k})$

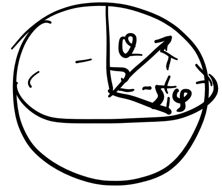
Classification of 2D class A strong TI
$= (\tilde{H}: S^2 \to C_0)$
$= \pi_2(C_0) = \mathbb{Z}$

Consider the simpler case with $m=n=1$ (2 bands):

$$C_0 = \frac{U(2)}{U(1)\times U(1)} \cong S^2$$

$$U(2) \ni U = \begin{pmatrix} a & b \\ -e^{i\theta} b^* & e^{i\theta} a \end{pmatrix}, \quad a, b \in \mathbb{C},\; |a|^2 + |b|^2 = 1$$

$$U \sim \begin{pmatrix} \cos\frac{\theta}{2} & e^{i\varphi}\sin\frac{\theta}{2} \\ -e^{-i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \quad \begin{matrix} 0 \le \theta \le \pi \\ 0 \le \varphi < 2\pi \end{matrix}$$



$$\pi_2\left(\frac{U(2)}{U(1)\times U(1)}\right) = \pi_2(S^2) = \mathbb{Z}.$$

$$[T^2, S^2] = \mathbb{Z}$$

For general $\pi_d \dfrac{U(m+n)}{U(m)\times U(n)}$:

① $\pi_d(X\times Y) \cong \pi_d(X)\times \pi_d(Y)$

$$S^d \to X\times Y \iff (S^d\to X,\ S^d \to Y)$$

$$S^d\times I \to X\times Y \iff (S^d\times I\to X,\ S^d\times I\to Y)$$

② Serre fibration : $F\to E\to B$

induces a long exact sequence :

$$\cdots \to \underbrace{\pi_{d+1}(B)}_{\pi_{d+1}} \to \underbrace{\pi_d(F)\to \pi_d(E)\to \pi_d(B)}_{\pi_d} \to \underbrace{\pi_{d-1}(F)\to\cdots}_{\pi_{d-1}}$$

$$\to \pi_{d+1}(F)\to \pi_{d+1}(E)\to \pi_{d+1}(B)\to$$

$$\to \pi_d(F)\to \pi_d(E)\to \pi_d(B)\to$$

$$\to \cdots$$

③ There is fibration :

$$U(n)\to U(n+1)\longrightarrow \frac{U(n+1)}{U(n)} = S^{2n+1} \qquad \text{for } n\geq 1.$$

$$U_{n\times n} \mapsto \left(\begin{array}{c|c} U_{n\times n} & 0 \\ \hline 0 & 1\end{array}\right)$$

$\dim = n^2$ $\quad \dim=(n+1)^2 \quad$ $\dim = (n+1)^2 - n^2 = 2n+1$

$$U_{n+1,n+1}\cdot z,\quad z\in\mathbb{C}^{n+1}\cong \mathbb{R}^{2n+2}$$

U is unitary $\Rightarrow (z^\dagger U^\dagger)(Uz) = z^\dagger z = 1$

$U(n+1)$ acts on S^{2n+1}

action of $U(n+1)$ on $\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\in\mathbb{C}^{n+1}$

$\in S^{2n+1}$

$$\left.\begin{array}{l} U\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \\ U\in U(n+1)\end{array}\right\} \Rightarrow U=\left(\begin{array}{c|c}1 & 0\\ \hline 0 & U_{n,n}\end{array}\right)$$

$$U(n+1)/U(n) \cong S^{2n+1}$$

$$\Rightarrow \cdots \to \underset{\substack{\| \, d+1<2n+1 \Leftrightarrow d<2n \\ 0}}{\pi_{d+1}(S^{2n+1})} \to \pi_d(U(n)) \to \pi_d(U(n+1)) \to \underset{\substack{\| \, d<2n+1 \\ 0}}{\pi_d(S^{2n+1})} \to \cdots$$

$$\Rightarrow \pi_d U(n) \cong \pi_d U(n+1) \quad \text{if } n>\frac{d}{2}$$

$$\pi_d U \cong \pi_d U(\infty)$$

$$\left\{\begin{array}{l}\pi_d\ U(n) \to 2 \\ \left.\begin{array}{l}\pi_d\ O(n) \\ \pi_d\ Sp(n)\end{array}\right\} \to 8\end{array}\right.$$

④ For $n>\frac{d}{2}$,

$$\pi_d U(n) = \begin{cases} 0, & \text{if } d \text{ even} \\ \mathbb{Z}, & \text{if } d \text{ odd}\end{cases} \qquad \text{(Bott periodicity)}$$

$\xrightarrow{\pi_d}$ $U(n)\downarrow$ $n>\frac{d}{2}$

G	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
$U(1)$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$U(2)$	“ $\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_3 \oplus \mathbb{Z}_{12}$	$\mathbb{Z}_2^{\oplus 2} \oplus \mathbb{Z}_{84}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$U(3)$	“ $\mathbb{Z}$	“ 0	“	0	$\mathbb{Z}$	$\mathbb{Z}_6$	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_3$	$\mathbb{Z}_{30}$	$\mathbb{Z}_4$	$\mathbb{Z}_{60}$	$\mathbb{Z}_6$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{84}$	$\mathbb{Z}_{36}$
$U(4)$	“ $\mathbb{Z}$	“ 0	“	“	“	0	$\mathbb{Z}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{120}$	$\mathbb{Z}_4$	$\mathbb{Z}_{60}$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{1680}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{72}$
$U(5)$	“ $\mathbb{Z}$	“ 0	“	“	“	“	“	0	$\mathbb{Z}$	$\mathbb{Z}_{120}$	0	$\mathbb{Z}_{360}$	$\mathbb{Z}_4$	$\mathbb{Z}_{1680}$	$\mathbb{Z}_6$
$U(6)$	“ $\mathbb{Z}$	“ 0	“	“	“	“	“	“	“	0	$\mathbb{Z}$	$\mathbb{Z}_{720}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{5040}$	$\mathbb{Z}_6$
$U(7)$	“	“	“	“	“	“	“	“	“	“	“	0	$\mathbb{Z}$	$\mathbb{Z}_{5040}$	0
$U(8)$	“	“	“	“	“	“	“	“	“	“	“	“	“	0	$\mathbb{Z}$

⑤ $\pi_d C_0 = \pi_d \dfrac{U(m+n)}{U(m)\times U(n)}$. (assume $m,n \gg d$)

$$U(m)\times U(n) \to U(m+n) \longrightarrow C_0 = \frac{U(m+n)}{U(m)\times U(n)}$$

$$\Rightarrow \underset{\substack{\| \\ 0}}{\pi_{2k} U(m+n)} \to \underset{\substack{\| \\ \mathbb{Z}}}{\pi_{2k} C_0} \to \underset{\substack{\| \\ \mathbb{Z}\times\mathbb{Z}}}{\pi_{2k-1} U(m)\times U(n)} \twoheadrightarrow \underset{\substack{\| \\ \mathbb{Z}}}{\pi_{2k-1} U(m+n)} \to \underset{\substack{\| \\ 0}}{\pi_{2k-1} C_0} \to \underset{\substack{\| \\ 0}}{\pi_{2k-2} U(m)\times U(n)}$$

$$\Rightarrow \pi_d C_0 = \begin{cases} \mathbb{Z}, & \text{if } d \text{ even} \\ 0, & \text{if } d \text{ odd.}\end{cases}$$

$\Rightarrow$ class A TI:

d	0	1	2	3	4	$\cdots$
classification	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$

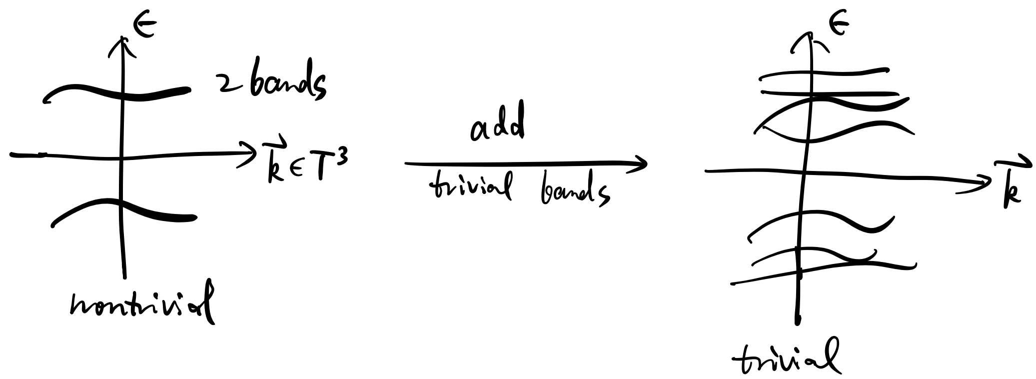
Remarks.

(1) Classification is obtained under stable condition

Example: 3D class A TI with 2 bands:

$$\tilde{H}: T^3 \to \frac{U(2)}{U(1)\times U(1)} = S^2$$

$$T^3 \to S^3 \xrightarrow[\mathbb{Z}]{\text{Hopf}} S^2$$



Adding bands.

$$H = H \oplus H_0$$

$$H_0 = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \quad X = U\begin{pmatrix} I_m & \\ & -I_n \end{pmatrix}U^\dagger$$

$$X^2 = I$$

Claim. $H_0 \sim H_1 = \begin{pmatrix} & iI \\ -iI & \end{pmatrix}$

homotopy $H_t = \cos\left(\frac{\pi}{2}t\right) H_0 + \sin\left(\frac{\pi}{2}t\right) H_1$

$$\left.\begin{aligned} H_0^2 &= (\sigma_z \otimes X)^2 = I_2 \otimes I = I \\ H_1^2 &= (-\sigma_y \otimes I)^2 = I \\ H_0 H_1 &= -H_1 H_0 \end{aligned}\right\} \Rightarrow H_t^2 = \left(\cos^2\frac{\pi}{2}t + \sin^2\frac{\pi}{2}t\right) I = I$$

$$\begin{pmatrix} X & \\ & -X \end{pmatrix} \underset{\text{gapped}}{\overset{\text{Symm.}}{\sim}} \begin{pmatrix} & iI \\ -iI & \end{pmatrix} \underset{\text{gapped}}{\overset{\text{Symm}}{\sim}} \begin{pmatrix} I & \\ & -I \end{pmatrix}$$

$\{X\}_{\text{Symm}}$ may have nontrivial topology, but $\left\{\begin{pmatrix} X & \\ & -X \end{pmatrix}\right\}_{\text{Symm.}}$ is always contractible.

Equivalence of modes (with different number of bands)

- $H_n' \sim H_n''$ iff $H_n' \oplus Y_k \sim H_n'' \oplus Y_k$ for some Y_k.

  "$\Leftarrow$": $H' \oplus Y \sim H'' \oplus Y \Rightarrow H' \oplus Y \oplus (-Y) \sim H'' \oplus \underbrace{Y \oplus (-Y)}$

  $\Rightarrow H' \sim H''$

- diffence class $d(A,B) \rightarrow$ understood as $A \ominus B$

  $$\underset{A' \ominus B'}{(A_n', B_n')} \underset{\sim}{\sim} \underset{A'' \ominus B''}{(A_m'', B_m'')} \text{ if } \underset{A' \oplus B''}{A_n' \oplus B_m''} \underset{\sim}{\sim} \underset{A'' \oplus B'}{A_m'' \oplus B_n'}$$

  This equi. gives a notion of equiv. of matrices with different sizes.

  $$(A_n, B_n) \sim \big(A_n \oplus (-B_n),\ B_n \oplus (-B_n)\big) = \left(A_n \oplus -B_n,\ \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}\right)$$

  We can always choose $B_n = \begin{pmatrix} I_s & \\ & -I_s \end{pmatrix}$ with $n = 2s$.

  Invariant of (A,B): $\quad k = k(A) - k(B) = k(A) - s$

  $k(A)$: # negative energies of A

- Consider again class A

  For each k, space of $\tilde{H}$ of class A is $\dfrac{U(2s)}{U(s+k) \times U(s-k)}$

  $$\text{“}\ C_0 = \frac{U(m+n)}{U(m) \times U(n)}\ \text{”} \longrightarrow C_0 = \bigcup_{k \in \mathbb{Z}} \lim_{s \to \infty} \frac{U(2s)}{U(s+k) \times U(s-k)}$$

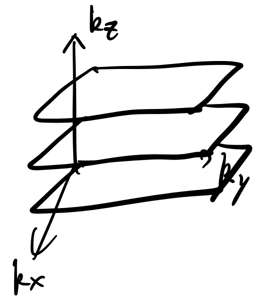
(2) strong / weak TI / TSC.

$$BZ^d = T^d \neq S^d$$

$$[T^d, C]_s \cong \pi_d(C) \oplus \bigoplus_{i=0}^{d-1} \binom{d}{i} \pi_i(C)$$

Strong TI

Weak TI
||
stacking of strong TI of lower dims.



$$H_{3D}(k_x,k_y,\cancel{k_z}) = H_{2D}(k_x,k_y)$$

(3) We can use topological invariants to distinguish TI/TSC classes for each symm class and dim.

(4) There are nontrivial (gapless, symmetric) edge states for nontrivial TI/TSC.

(5) Space of Hamiltonians for each symm class.

symmetric
No gapped condition

$\left\{ \begin{array}{l} \text{① gapped} \\ \text{② flatten } \epsilon = \pm 1 \end{array} \right. \Leftrightarrow \tilde{H}^2 = 1$

	class	TCS	{Hamiltonian H}	{simplified Hamiltonian $\tilde{H}$}
(complex)	A	000	$u(N)$	$U(m+n)/U(m)\times U(n) = C_0$
	AIII	001	$u(M+N)/u(M)\times u(N)$	$u(N) = C_1$
(real)	AI	+00	$u(N)/o(N)$	$O(m+n)/O(m)\times O(n) = R_0$
	BDI	++0	$o(M+N)/o(M)\times o(N)$	$O(n) = R_1$
	D	0+0	$o(N)$	$O(2n)/U(n) = R_2$
	DIII	−+1	$so(2N)/u(N)$	$U(2n)/Sp(n) = R_3$
	AII	−00	$u(2N)/sp(N)$	$Sp(m+n)/Sp(m)\times Sp(n) = R_4$
	CII	−−1	$sp(M+N)/sp(M)\times sp(N)$	$Sp(n) = R_5$
	C	0−0	$sp(N)$	$Sp(n)/U(n) = R_6$
	CI	+−1	$sp(N)/u(N)$	$U(n)/O(n) = R_7$

complex: d-dim strong TI of class A or AIII are classified by $\pi_d(C_i)$.

real: $\begin{cases} \text{0-dim strong TI/TSC are classified by } \pi_0(R_i). \\ \text{d-dim } (d\geq 1) \quad \dots \quad \dots \quad \dots \quad [T^d, R_i] \end{cases}$

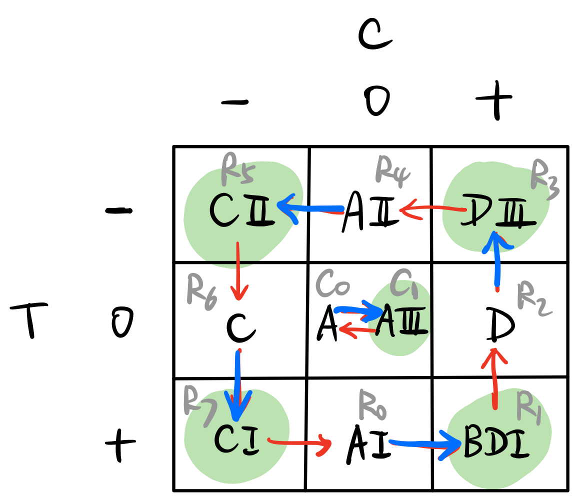
T: $$U_T H^* U_T^\dagger = H$$

$$U_T H(\vec{k})^* U_T^\dagger = H(-\vec{k})$$

If $\vec{k} = -\vec{k} \in T^d$, then $H(\vec{k}) \in R_i$

7.4. Shift of dimensions and symmetries

Cartan\\d	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$



chiral symmetry

(1) →: removing chiral symmetry $\begin{cases} C_1 \to C_0 \\ R_{2k+1} \to R_{2k+2 \,(\mathrm{mod}\, 8)} \end{cases}$

(2) →: adding chiral symmetry. $\begin{cases} C_0 \to C_1 \\ R_{2k} \to R_{2k+1 \,(\mathrm{mod}\, 8)} \end{cases}$

(1) d-dim chiral symm $\to$ (d+1)-dim without chiral symm.

chiral symm: $U_s H(\vec{k_d}) U_s^\dagger = -H(\vec{k_d}) \Longleftrightarrow \{U_s, H(\vec{k_d})\} = 0$

Construct d+1 dim model

$$H'(\vec{k_d}, k_{d+1}) := H(\vec{k_d}) \cos k_{d+1} + U_s \sin k_{d+1}$$

$\Rightarrow$ eigenvalues: $\pm\sqrt{\varepsilon(\vec{k_d})^2 \cos^2 k_{d+1} + \sin^2 k_{d+1}} \xlongequal{\varepsilon=\pm 1} \pm 1$

H' does NOT has chiral symm U_s.

① complex class AIII $\to$ A.

② real classes $\{DIII, CII, CI, BDI\}_{s=1} \xrightarrow{?} \{AII, C, AI, D\}_{s=0}$

time reversal $\begin{cases} U_T H(\vec{k_d})^* U_T^\dagger = H(-\vec{k_d}) \\ U_T U_T^* = \xi_T = \pm 1 \\ U_T U_T^\dagger = 1 \end{cases}$ $\Rightarrow (U_T)^T = \xi_T U_T$

charge conj. $\begin{cases} U_c H(\vec{k_d})^* U_c^\dagger = -H(-\vec{k_d}) \\ U_c U_c^* = \xi_c = \pm 1 \\ U_c U_c^\dagger = 1 \end{cases}$ $\Rightarrow (U_c)^T = \xi_c U_c$

chiral symm $\begin{cases} U_s H(\vec{k_d}) U_s^\dagger = -H(\vec{k_d}) \\ U_s = U_T \cdot U_c^* \\ U_s^2 = 1 \end{cases}$ $\Rightarrow U_T U_c^* U_T U_c^* = 1 \Rightarrow U_c^* U_T = \xi_c \xi_T U_T^* U_c$

Q: symm of $H'(\vec{k_d}, k_{d+1}) = H(\vec{k_d}) \cos k_{d+1} + U_s \sin k_{d+1}$?

A: $U_T \cdot H'(\vec{k_d}, k_{d+1})^* \cdot U_T^\dagger = U_T H(\vec{k_d})^* U_T^\dagger \cos k_{d+1} + U_T U_s^* U_T^\dagger \sin k_{d+1}$

$= H(-\vec{k_d}) \cos k_{d+1} + U_T U_s^* U_T^\dagger \sin k_{d+1}$

$\overset{?}{=} H'(-\vec{k_d}, -k_{d+1})$

$= H(-\vec{k_d}) \cos(-k_{d+1}) + U_s \sin(-k_{d+1})$

$\Rightarrow H'$ has U_T symm iff $U_T U_s^* U_T^\dagger = -U_s$

$\Leftrightarrow U_T (U_T U_c^*)^* U_T^\dagger = -U_T U_c^*$

$\Leftrightarrow \quad U_c U_T^* = -U_T U_c^*$

$\Leftrightarrow \quad \xi_c \xi_T = -1$

$\Rightarrow \begin{cases} \text{DIII} \to \text{AII} \\ \text{CI} \to \text{AI} \end{cases}$

$$U_c H'(\vec{k}_d, k_{d+1})^* U_c^\dagger = U_c H(\vec{k}_d) U_c^\dagger \cos k_{d+1} + U_c U_s^* U_c^\dagger \sin k_{d+1}$$
$$\overset{?}{=} -H'(-\vec{k}_d, -k_{d+1})$$

$\Rightarrow$ H' has U_c symm iff $U_c U_s^* U_c^\dagger = U_s$

$\Leftrightarrow \quad \xi_c \cdot \xi_T = +1$

$\Rightarrow \begin{cases} \text{CII} \to \text{C} \\ \text{BDI} \to \text{D} \end{cases}$

(2) d-dim, No chiral symm $\to$ (d+1)-dim, chiral symm.

$$U_s H(\vec{k}_d) U_s^\dagger = -H(\vec{k}_d) \Leftrightarrow \{U_s, H(\vec{k}_d)\} = 0$$

Construct d+1 dim model with band number doubled:

$$H'(\vec{k}_d, k_{d+1}) = \underbrace{H(\vec{k}_d)\otimes\sigma_x}\cos k_{d+1} + \underbrace{I\otimes\sigma_y}\sin k_{d+1}$$

anticommute

Eigenvalues of H': $\pm\sqrt{\varepsilon(\vec{k}_d)^2\cos^2 k_{d+1} + \sin^2 k_{d+1}} \overset{\varepsilon=\pm1}{=\!=} \pm 1$

H' has chiral symmetry $U_s = I\otimes\sigma_z$: $U_s H'(k) U_s^\dagger = -H'(k)$

① Complex class $\underset{(000)}{\text{A}} \to \underset{(001)}{\text{AIII}}$.

② Real classes $\{\text{D}, \text{AII}, \text{C}, \text{AI}\}_{s=0} \to \{\text{DIII}, \text{CII}, \text{CI}, \text{BDI}\}_{s=1}$

• If $\exists T$ symm, $U_T H(\vec{k}_d)^* U_T^\dagger = H(-\vec{k}_d)$ with $U_T U_T^* = \xi_T$, then

$$\left.\begin{aligned} &(U_T\otimes I)\cdot H'(\vec{k}_d, k_{d+1})^* \cdot (U_T\otimes I)^\dagger = H'(-\vec{k}_d, -k_{d+1}) \\ &\xi_T' = (U_T\otimes I)\cdot(U_T\otimes I)^* = \xi_T \end{aligned}\right\} \Rightarrow H' \text{ has } U_T\otimes I \text{ symm.} \quad (\xi_T' = \xi_T)$$

Consider charge conjugation symmetry U_c':

$$U_s' = U_T' U_c'^* \Rightarrow U_c' = (U_T'^\dagger U_s')^* = U_T'^T U_s'^*$$

$$\left.\begin{aligned} U_S' &= I\otimes\sigma_z \\ U_T' &= U_T\otimes I \end{aligned}\right\}$$

$\Rightarrow U_C' = (U_T^T\otimes I)\cdot(I\otimes\sigma_z) = \xi_T\, U_T\otimes\sigma_z$

$\Rightarrow \xi_C' = U_C' U_C'^* = (U_T\otimes\sigma_z)\cdot(U_T^*\otimes\sigma_z) = \xi_T$

$\Rightarrow (\xi_T, 0, 0) \to (\xi_T, \xi_T, 1)$

$\Rightarrow \begin{cases} \mathrm{AII} \to \mathrm{CII} \\ \mathrm{AI} \to \mathrm{BDI} \end{cases}$

- If $\exists$ C symm, $U_C H(\vec{k_d})^* U_C^\dagger = -H(-\vec{k_d})$ with $U_C U_C^* = \xi_C$, then

$$\left.\begin{aligned} &(U_C\otimes\sigma_x)\cdot H'(\vec{k_d}, k_{d+1})^*\cdot(U_C\otimes\sigma_x)^\dagger = -H'(-\vec{k_d}, -k_{d+1}) \\ &\xi_C' = (U_C\otimes\sigma_x)\cdot(U_C\otimes\sigma_x)^* = \xi_C \end{aligned}\right\} H' \text{ has } U_C\otimes\sigma_x \text{ symm. } (\xi_C' = \xi_C)$$

Consider time reversal symm U_T':

$$U_S' = U_T' U_C'^* \Rightarrow \left.\begin{aligned} U_T' &= U_S' U_C'^T \\ U_S' &= I\otimes\sigma_z \\ U_C' &= U_C\otimes\sigma_x \end{aligned}\right\}$$

$\Rightarrow U_T' = (I\otimes\sigma_z)\cdot(U_C^T\otimes\sigma_x) = U_C^T\otimes(i\sigma_y) = \xi_C\; U_C\otimes(i\sigma_y)$

$\Rightarrow \xi_T' = U_T' U_T'^* = \xi_C(U_C\otimes i\sigma_y)\cdot\xi_C(U_C^*\otimes i\sigma_y) = -\xi_C$

$\Rightarrow (0, \xi_C, 0) \longrightarrow (-\xi_C, \xi_C, 0)$

$\Rightarrow \begin{cases} \mathrm{C} \to \mathrm{CI} \\ \mathrm{D} \to \mathrm{DIII} \end{cases}$