6. Examples of TI/TSC

Chern insulator is protected by $U(1)_c$ charge conservation symmetry.

Fermion parity $\mathbb{Z}_2^f = \{1, P_f = (-1)^F\}$

$F = \sum_j c_j^\dagger c_j$ is the fermion number.

$$H = \sum c^\dagger c + (\Delta cc + h.c.) + V c^\dagger c^\dagger cc + \cdots$$ preserves $\mathbb{Z}_2^f$.

6.1. 1+1D Majorana chain (by Kitaev 2000)

- Majorana fermion = real fermion

complex fermion $c, c^\dagger$:

$$\begin{cases} (c^\dagger)^2 = c^2 = 0 \\ cc^\dagger + c^\dagger c = 1 \end{cases}$$

Split c and $c^\dagger$ into "real" part and "imaginary" part:

$$\begin{cases} c = \frac{1}{2}(\gamma_1 + i\gamma_2) \\ c^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2) \end{cases} \qquad \begin{cases} \gamma_1 = c + c^\dagger \\ \gamma_2 = \frac{1}{i}(c - c^\dagger) \end{cases}$$

Satisfying:

$$\begin{cases} \gamma_1^\dagger = \gamma_1, \quad \gamma_2^\dagger = \gamma_2 \\ \gamma_1^2 = (c + c^\dagger)^2 = cc^\dagger + c^\dagger c = 1 \\ \gamma_2^2 = \frac{1}{i^2}(c - c^\dagger)^2 = cc^\dagger + c^\dagger c = 1 \\ \gamma_1\gamma_2 + \gamma_2\gamma_1 = \frac{1}{i}(c + c^\dagger)(c - c^\dagger) + \frac{1}{i}(c - c^\dagger)(c + c^\dagger) = 0 \end{cases}$$

$$\begin{cases} H = \epsilon c^\dagger c \Rightarrow \begin{cases} E = \epsilon, \ |ES\rangle = |n=1\rangle, \quad n = c^\dagger c = 1 \\ E = 0, \ |GS\rangle = |n=0\rangle, \quad n = c^\dagger c = 0 \end{cases} \\ \begin{cases} n = c^\dagger c = \frac{1}{4}(\gamma_1 - i\gamma_2)(\gamma_1 + i\gamma_2) = \frac{1}{4}(2 + 2i\gamma_1\gamma_2) = \frac{1}{2}(1 + i\gamma_1\gamma_2) \\ P_f = (-1)^n = 1 - 2n = -i\gamma_1\gamma_2 = \pm 1 \end{cases} \\ H = \epsilon n = \epsilon \cdot \frac{1 - P_f}{2} = \frac{\epsilon}{2}(1 + \underbrace{i\gamma_1\gamma_2}) \end{cases}$$

2 Majorana fermions $\gamma_1, \gamma_2 \longleftrightarrow$ 1 complex fermion mode c

$\gamma_1 \longrightarrow \gamma_2$ $(-i\gamma_1\gamma_2 = 1)$ $\quad P_f = -i\gamma_1\gamma_2 = 1 \quad n = c^\dagger c = 0$

$\gamma_1 \longleftarrow \gamma_2$ $(-i\gamma_2\gamma_1 = 1)$ $\quad P_f = -i\gamma_1\gamma_2 = -1 \quad n = c^\dagger c = 1$

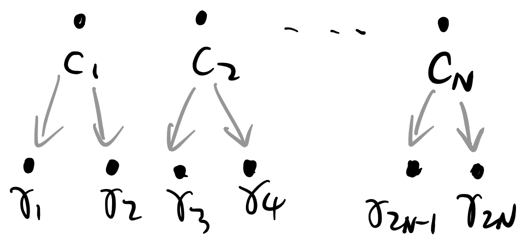
change pairing direction $\Longleftrightarrow$ change fermion parity $P_f = \pm 1$

- Majorana chain.

Consider N complex fermions $=$ $2N$ Majorana fermions.

$$c_j = \frac{1}{2}(\gamma_{2j-1} + i\gamma_{2j})$$

$(j = 1, 2, \cdots, N)$



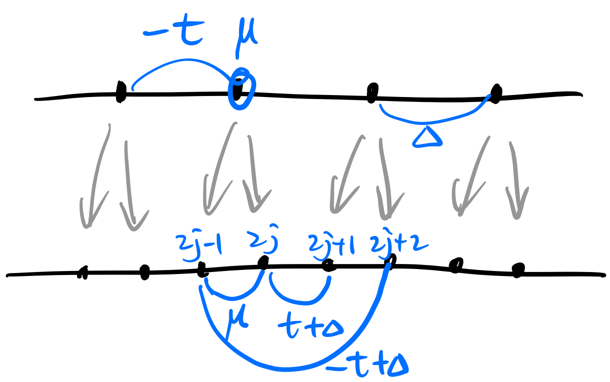
(1) p-wave superconductor chain.

$$H = \sum_j \left[-t(c_j^\dagger c_{j+1} + h.c.) - \mu c_j^\dagger c_j + \Delta(c_j c_{j+1} + h.c.) \right]$$

periodic boundary $\rightarrow$ $-t(c_N^\dagger c_1 + h.c.)$ $\qquad t, \mu, \Delta \in \mathbb{R}.$

$$= \sum_j \left[-t \frac{1}{4}(\gamma_{2j-1} - i\gamma_{2j})(\gamma_{2j+1} + i\gamma_{2j+2}) + h.c. + \cdots \right]$$

$$= \frac{i}{2} \sum_j \left[-\mu \gamma_{2j-1}\gamma_{2j} + (t+\Delta)\gamma_{2j}\gamma_{2j+1} + (-t+\Delta)\gamma_{2j-1}\gamma_{2j+2} \right]$$



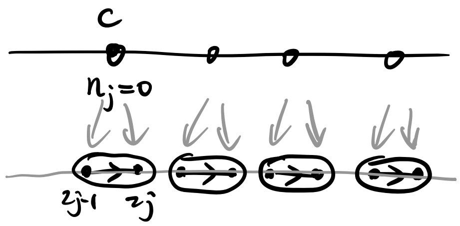
$$= \begin{cases} (\mu = -2, t = \Delta = 0) = \sum_j i\gamma_{2j-1}\gamma_{2j} = -\sum_j (P_f)_j \\ (\mu = 0, t = \Delta = 1) = \sum_j i\gamma_{2j}\gamma_{2j+1} \end{cases}$$

(2) Trivial chain $(\mu = -2, t = \Delta = 0)$

$$H = -\sum_j (P_f)_j$$

GS : $n_j = 0 \Leftrightarrow (P_f)_j = +1$

$\Leftrightarrow -i\gamma_{2j-1}\gamma_{2j} = 1$

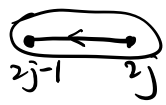


$|GS\rangle = \bigotimes_j |n_j = 0\rangle$

$|GS\rangle = \bigotimes_j |-i\gamma_{2j-1}\gamma_{2j} = 1\rangle$

$-i\gamma_{2j-1}\gamma_{2j} = 1$

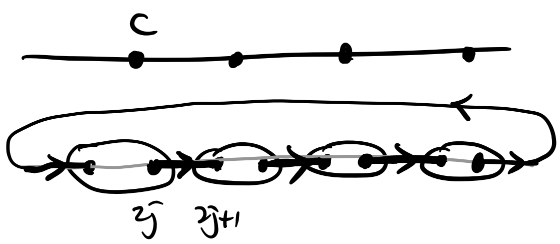
ES : $n_j = 1$ for some j.



(3) Nontrivial chain ($\mu = 0$, $t = \Delta = 1$)

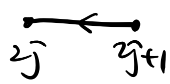
$$H = -\sum_j (c_j^\dagger c_{j+1} + h.c.) + \sum_j (c_j c_{j+1} + h.c.)$$

$$= \sum_j i\gamma_{2j}\gamma_{2j+1}$$



$|GS\rangle = \bigotimes_j |-i\gamma_{2j}\gamma_{2j+1} = 1\rangle$

excited state :



$-i\gamma_{2j}\gamma_{2j+1} = -1$.

Fermion parity of $|GS\rangle$.

$$P_f|GS\rangle = \prod_j (-i\gamma_{2j-1}\gamma_{2j})|GS\rangle$$

$$= (-i)^N (\gamma_1\gamma_2)(\gamma_3\gamma_4)\cdots(\gamma_{2N-1}\gamma_{2N})\,|GS\rangle$$

$$= (-i)^N \gamma_1 (\gamma_2\gamma_3)(\gamma_4\gamma_5)\cdots(\gamma_{2N-2}\gamma_{2N-1})\,\gamma_{2N}\,|GS\rangle$$

$$= (-i)^N (\gamma_2\gamma_3)\cdots(\gamma_{2N-2}\gamma_{2N-1})(\gamma_1\gamma_{2N})\,|GS\rangle$$

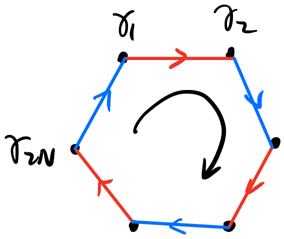
$$= -i\gamma_1\gamma_{2N}\,|GS\rangle$$

$$= -(-i\gamma_{2N}\gamma_1)\,|GS\rangle$$

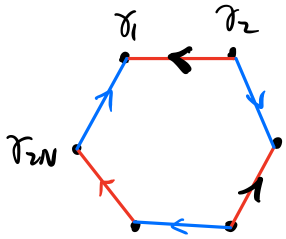
$$= -|GS\rangle$$

$P_f = -1$ acting on $|GS\rangle$

$\Leftrightarrow$ $|GS\rangle$ has odd number of complex fermions c_j.



$$\begin{cases} |\psi_1\rangle : \ -i\gamma_1\gamma_2 = -i\gamma_3\gamma_4 = \cdots = -i\gamma_{2N-1}\gamma_{2N} = 1 \\ |\psi_2\rangle : \ -i\gamma_2\gamma_3 = -i\gamma_4\gamma_5 = \cdots = -i\gamma_{2N}\gamma_1 = 1 \\ |\psi_1\rangle \text{ and } |\psi_2\rangle \text{ have different fermion parity.} \end{cases}$$

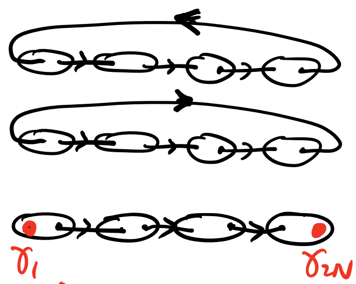


$|\psi_1\rangle$ and $|\psi_2\rangle$ have $\begin{cases} \text{same} \\ \text{different} \end{cases}$ fermion parity

$\Leftrightarrow$ number of counter clockwise arrow N_{AC} is $\begin{cases} \text{odd} \\ \text{even} \end{cases}$

$\Leftrightarrow$ loop is $\begin{cases} \text{Kastelyn} \\ \text{non-Kastelyn} \end{cases}$ oriented

$$\begin{cases} \text{periodic boundary condition} \\ \text{antiperiodic} \quad \cdots \\ \text{open} \quad \cdots\cdots \end{cases}$$



$P_f = -1$

$P_f = 1$

edge state

γ_1, γ_{2N} are unpaired Majorana fermion.

GSD = 2

- p-wave SC in momentum space.

$$H = \sum_j \left[-t\left(c_j^\dagger c_{j+1} + h.c.\right) - \mu c_j^\dagger c_j + \Delta\left(c_j c_{j+1} + h.c.\right) \right]$$

$$\begin{cases} c_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} c_k \\ c_j^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ikj} c_k^\dagger \end{cases}$$

$$= \sum_k \left[-t\left(e^{ik} c_k^\dagger c_k + h.c.\right) - \mu c_k^\dagger c_k + \Delta\underbrace{\left(e^{ik} c_{-k} c_k + h.c.\right)}_{} \right]$$

$$\sum_k \Delta e^{ik} c_{-k} c_k$$

$$= \frac{1}{2}\sum_k \Delta e^{ik} c_{-k} c_k + \frac{1}{2}\sum_k \Delta e^{-ik} c_{-k} c_k$$

$$= \frac{1}{2}\sum_k \left(\Delta\, e^{ik} c_{-k} c_k - \Delta e^{-ik} c_{-k} c_k\right)$$
$$= \sum_k i\Delta \sin k\; c_{-k} c_k$$

$$= \sum_k \left(-2t\cos k - \mu\right) c_k^\dagger c_k + \Delta\left(i \sin k\; c_{-k} c_k + h.c.\right)$$

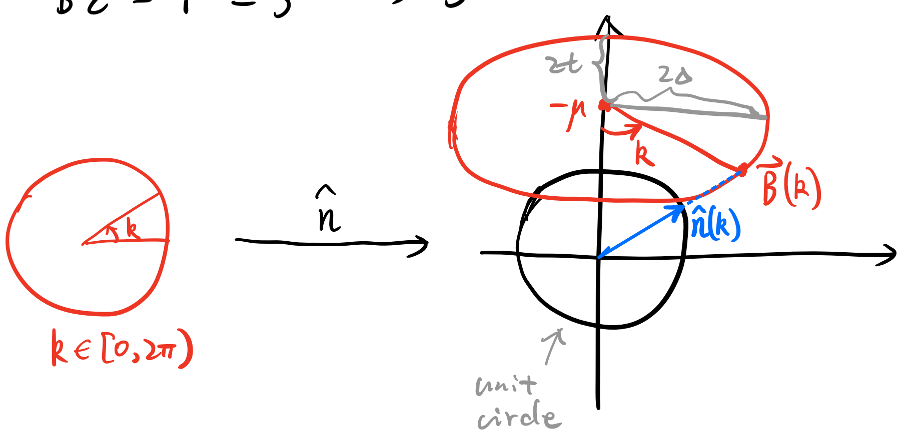
$$= \sum_k \frac{1}{2} \left(c_k^\dagger, c_{-k}\right) \underbrace{\begin{pmatrix} -2t\cos k - \mu & -i\Delta \sin k \\ 2i\Delta \sin k & 2t\cos k + \mu \end{pmatrix}}_{\mathcal{H}_k} \underbrace{\begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}}_{\text{Nambu basis}}$$

$$\mathcal{H}_k = 2\Delta \sin k\; \sigma_y - (2t\cos k + \mu)\, \sigma_z$$
$$= \vec{B}(k)\cdot\vec{\sigma} = E(k)\; \hat{n}(k)\cdot\vec{\sigma}$$

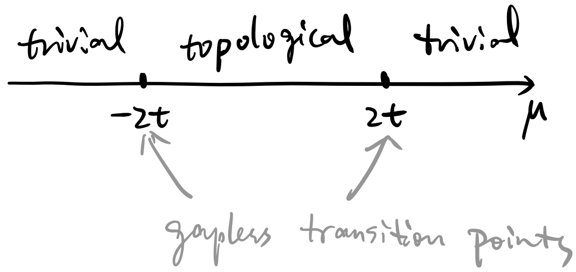
$$\vec{B}(k) := \left(2\Delta \sin k,\; -2t\cos k - \mu\right)$$

$$\hat{n}(k) := \frac{\vec{B}(k)}{|\vec{B}(k)|}$$

$$\hat{n}: \quad BZ = T^1 = S^1 \longrightarrow S^1$$



$$\text{mapping degree} = \text{winding number} = \begin{cases} 1, & \text{if } -\mu - 2t < 0 < -\mu + 2t \\ 0, & \text{otherwise} \end{cases}$$



phase diagram

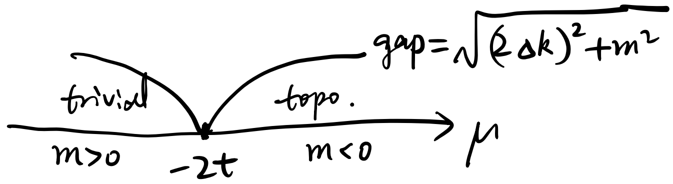
- Continuum model, field theory.

Consider $\mu = -2t - m$ with m small.

$$\mathcal{H}_k = 2\Delta \sin k\; \sigma_y + (-2t\cos k + 2t + m)\, \sigma_z$$

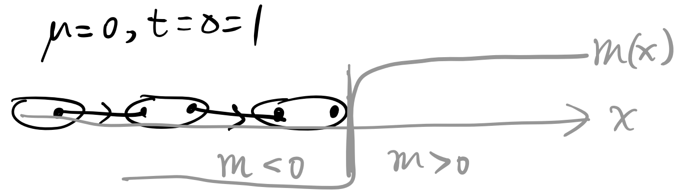
$$\xrightarrow{k\to 0} 2\Delta k\,\sigma_y + m\,\sigma_z$$

1+1 D massive Dirac Hamiltonian.



- Majorana edge mode

Consider $\mu=0,\ t=\Delta=1$



domain wall for mass $m(x)$.

For $\mu=-2t-m\approx -2t$:

$$\mathcal{H}_k = 2\Delta\cdot k\,\sigma_y + m\,\sigma_z$$

$$\downarrow$$

$$\mathcal{H} = 2\Delta\,\sigma_y\, i\partial_x + m(x)\,\sigma_z$$

Try to solve zero energy state near $x=0$:

$$\mathcal{H}\,\Psi(x)=0$$

$$\Leftrightarrow\ 2\Delta\,\sigma_y\, i\partial_x \Psi(x) + m(x)\,\sigma_z\,\Psi(x)=0 \qquad v\sim\frac{\Delta}{m}$$

$$\Leftrightarrow\ \partial_x\Psi(x) = -i\,\frac{1}{v}\,m(x)\,\sigma_x\,\Psi(x)$$

choose $\Psi(x)=\varphi(x)\begin{pmatrix}1\\ \pm 1\end{pmatrix}$

$$\Leftrightarrow\ \partial_x\varphi(x) = \pm\frac{1}{v}\,m(x)\,\varphi(x)$$

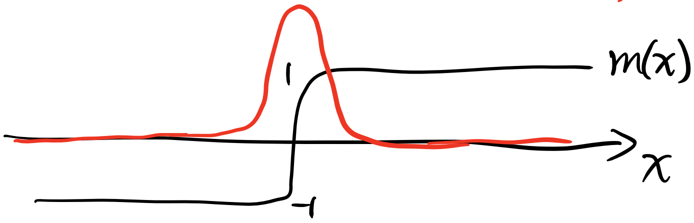
$$\Leftrightarrow\ \varphi(x) = e^{\pm\frac{1}{v}\int_0^x m(x')\,dx'}$$

$$\Leftrightarrow\ \Psi(x) = e^{\pm\frac{1}{v}\int_0^x m(x')\,dx'}\begin{pmatrix}1\\ \pm 1\end{pmatrix}$$

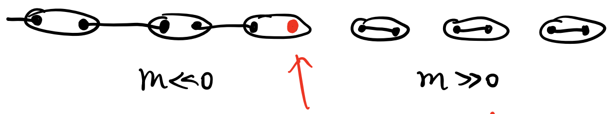
$$\begin{cases} ①\ \Psi_+(x) = e^{+\frac{1}{v}\int_0^x m(x')\,dx'}\begin{pmatrix}1\\ 1\end{pmatrix} \xrightarrow[x\to-\infty]{x\to\infty} e^{\frac{1}{v}x}\begin{pmatrix}1\\ 1\end{pmatrix} & \times \\ ②\ \Psi_-(x) = e^{-\frac{1}{v}\int_0^x m(x')\,dx'}\begin{pmatrix}1\\ -1\end{pmatrix} \xrightarrow{x\to\pm\infty} e^{-\frac{1}{v}x}\begin{pmatrix}1\\ -1\end{pmatrix} & \checkmark \end{cases}$$

$l=v\sim\frac{\Delta}{m}$

$\Psi_-(x) = e^{-\frac{1}{v}|x|}\begin{pmatrix}1\\-1\end{pmatrix}$: zero-energy edge mode.



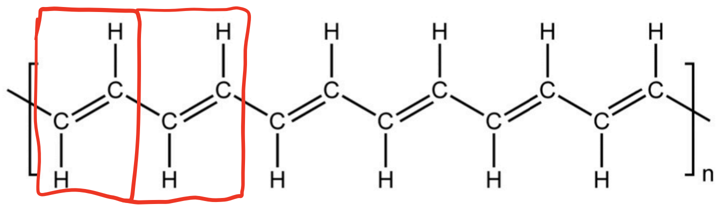
domain wall fermion

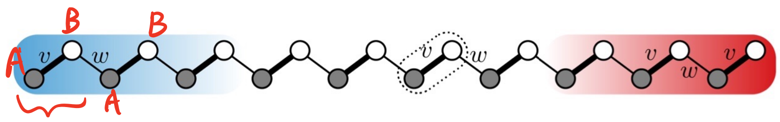


fully localized $\ell \sim 0 \sim a$ (lattice const)

6.2. 1D SSH model

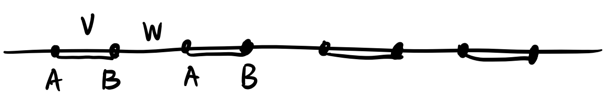
SSH = Su-Schrieffer-Heeger





unit cell

2 sites / unit cell



$$H = \sum_i \left(v\, c_{iA}^\dagger c_{iB} + w\, c_{iB}^\dagger c_{i+1A} + h.c. \right)$$

$$= \sum_k \left(c_{kA}^\dagger, c_{kB}^\dagger \right) \underbrace{\begin{pmatrix} 0 & v+we^{-ik} \\ v+we^{ik} & 0 \end{pmatrix}}_{\mathcal{H}_k} \begin{pmatrix} c_{kA} \\ c_{kB} \end{pmatrix}$$

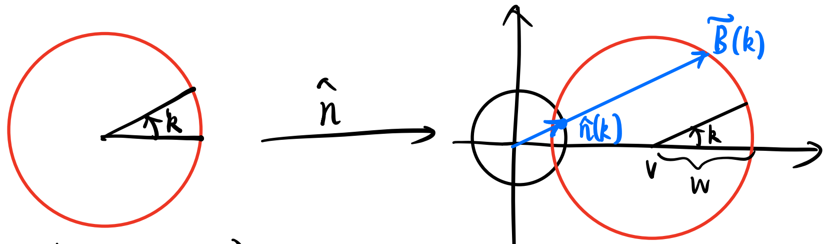
$$\mathcal{H}_k = (v + w\cos k)\,\sigma_x + w \sin k\, \sigma_y$$

$$= \vec{B}(k) \cdot \vec{\sigma}$$

$$= E(k) \quad \hat{n}(k) \cdot \vec{\sigma}$$

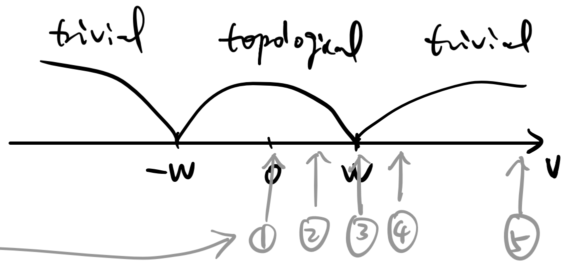
$\vec{B}(k) = (v + w\cos k,\ w \sin k)$ $\qquad E(k) := |\vec{B}(k)| = \sqrt{v^2 + w^2 + 2vw\cos k}$

$\hat{n}(k) := \vec{B}(k) / |\vec{B}(k)|$



$k \in BZ = [0, 2\pi)$

mapping degree = winding number = $\begin{cases} 1, & \text{if } v - w < 0 < v + w \\ 0, & \text{otherwise} \end{cases}$



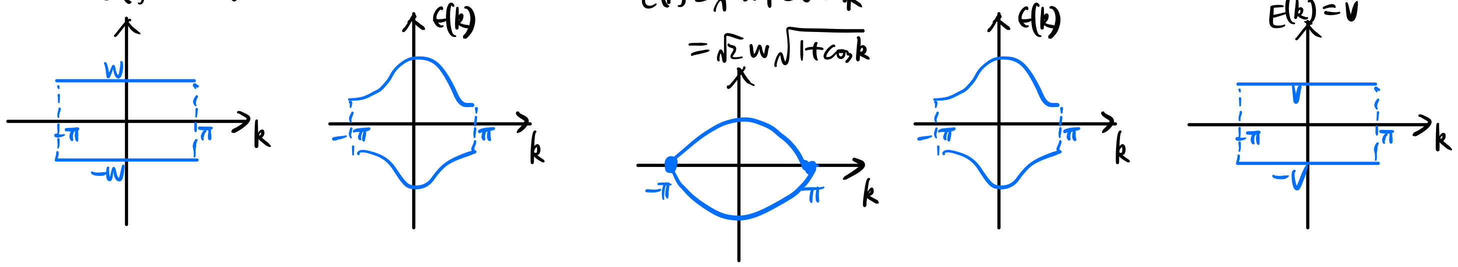
① $v = 0$, $\epsilon(k) = \pm E(k) = \pm w$

② $0 < v < w$

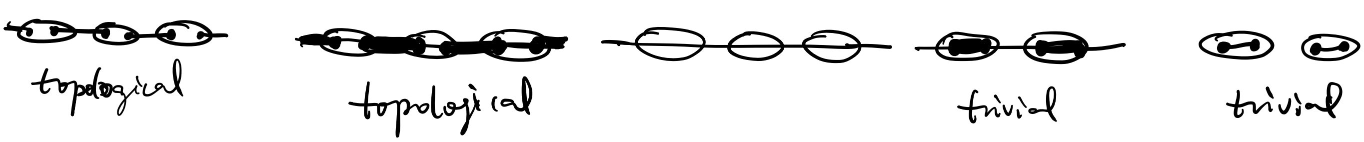
③ $v = w$ $E(k) = \sqrt{2w^2 + 2w^2 \cos k}$ $= \sqrt{2}\, w \sqrt{1 + \cos k}$

④ $0 < v < w$

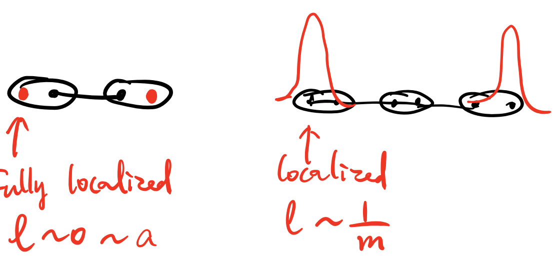
⑤ $v \gg w$ $E(k) = v$



bulk:



edge:



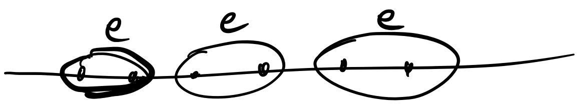
no edge states.

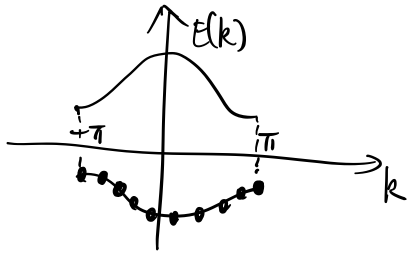
Similar to 1D Majorana chain (p-wave SC)!

Differences:

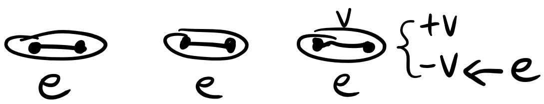
Symm.: $\begin{cases} \text{Majorana chain has } \mathbb{Z}_2^f \text{ symmetry.} \\ \text{SSH has } U(1)_f \supset \mathbb{Z}_2^f \text{ symmetry.} \end{cases}$

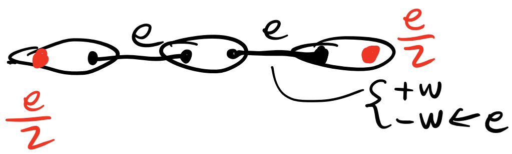
edge: $\begin{cases} \text{unpaired Majorana fermion} \sim \text{half complex fermion} \sim \text{half } \mathbb{Z}_2^f \text{ charge} \\ \text{SSH edge state} \sim \text{half } U(1)_f \text{ charge.} \end{cases}$





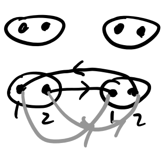
N unit cell
N electrons } ⇒ insulator

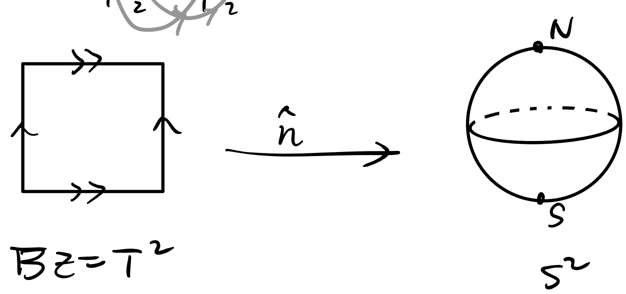




6.3. Another model for 2D Chern insulator.

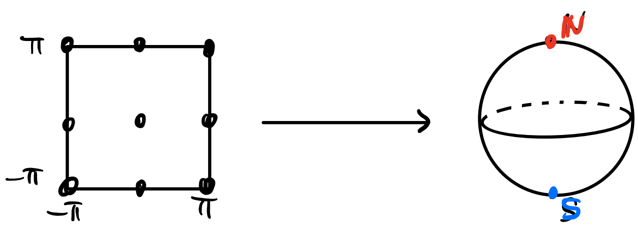
$$\mathcal{H}_{\vec{k}} = \sin k_x \, \sigma_x + \sin k_y \, \sigma_y + (u + \cos k_x + \cos k_y)\, \sigma_z$$
$$= \vec{B}(\vec{k}) \cdot \vec{\sigma}$$





$$C = \text{mapping degree} = \sum_{\vec{k}_0 \in \hat{n}^{-1}\binom{N}{S}} \operatorname{sgn} \det D\left(\hat{n}(\vec{k}_0)\right)$$

$$\hat{n}(\vec{k}_0) = \begin{cases} N \\ S \end{cases} = \begin{cases} (0,0,1) \\ (0,0,-1) \end{cases} \Longleftrightarrow \begin{cases} \sin k_x = 0 \Leftrightarrow k_x = 0, \pi \\ \sin k_y = 0 \Leftrightarrow k_y = 0, \pi \\ u + \underbrace{\cos k_x + \cos k_y}_{[-2,2]} \quad \begin{matrix} >0 \\ <0 \end{matrix} \end{cases}$$



$u<-2$: $(k_x,k_y)=\left(\begin{smallmatrix}0\\\pi\end{smallmatrix},\begin{smallmatrix}0\\\pi\end{smallmatrix}\right)\to S$

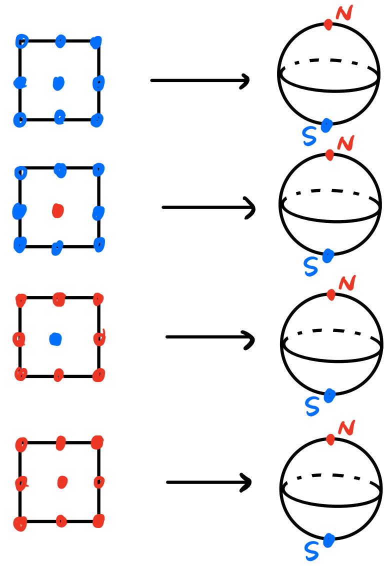
$C=0$

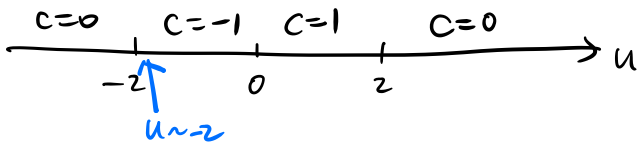
$-2<u<0$:

$C=-1$

$0<u<2$: $C=+1$

$u>2$: $c=0$





Edge state of the Chern insulator :

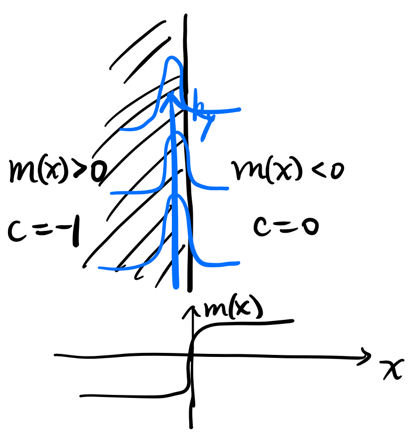
At $u\sim-2$, $\vec{k}\sim(0,0)$

$$\mathcal{H}_{\vec{k}}=\sin k_x\,\sigma_x+\sin k_y\,\sigma_y+(u+\cos k_x+\cos k_y)\,\sigma_z$$

$$\sim k_x\,\sigma_x+k_y\,\sigma_y+\underbrace{(u+1+1)}_{m\,\sim\,0}\,\sigma_z$$

$$=k_x\sigma_x+k_y\sigma_y+m\sigma_z$$ → 2+1D Dirac Hamiltonian.

$$\begin{cases} m<0: & c=0 \quad \text{trivial} \\ m>0: & c=-1 \quad \text{topological} \end{cases}$$



$$\mathcal{H}(x,k_y)=-i\,\sigma_x\partial_x+k_y\,\sigma_y+m(x)\,\sigma_z$$

Consider first $k_y = 0$ zero energy state :

$$\mathcal{H}(x, k_y=0) = -i\,\sigma_x \partial_x + m(x)\,\sigma_z$$

(1D Majorana chain edge problem)

$$\Rightarrow -i\sigma_x \partial_x \Psi_{k_y=0}(x) = -m(x)\,\sigma_x\, \Psi_{k_y=0}(x)$$

$$\Rightarrow \partial_x \Psi_{k_y=0}(x) = -m(x)\,\sigma_y\, \Psi_{k_y=0}(x)$$

$$\Rightarrow \Psi_{k_y=0}(x) = e^{-|x|}\begin{pmatrix}1\\ i\end{pmatrix}$$

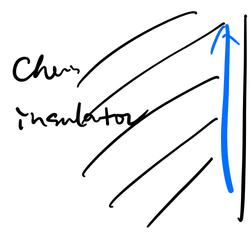
Consider k_y :

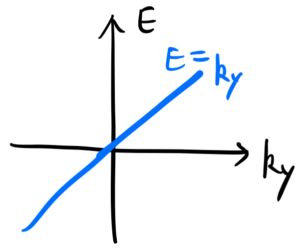
$$\Psi_{k_y}(x) = e^{-|x|}\begin{pmatrix}1\\ i\end{pmatrix}$$

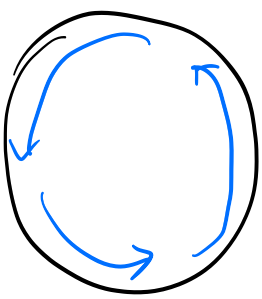
$$\Leftrightarrow \Psi(x,y) = \underbrace{e^{-|x|}}_{x}\,\underbrace{e^{ik_y\cdot y}}_{y}\begin{pmatrix}1\\ i\end{pmatrix} \quad \text{decoupled}$$

$$\Rightarrow \mathcal{H}_{k_y}(x)\,\Psi(x,y) = k_y\,\sigma_y\,\Psi(x,y) = k_y\,\Psi(x,y)$$

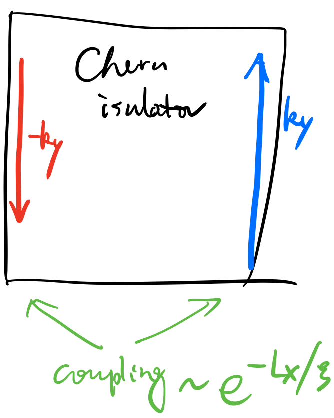
$$\Rightarrow \Psi(x,y) \text{ is eigenvector of } \mathcal{H} \text{ with } E_{\vec{k}} = k_y$$

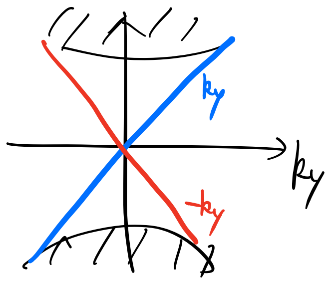






1+1D chiral fermion





6.4. 2D quantum spin Hall effect.

- Time reversal symmetry.

$$\begin{cases} \mathcal{T}\hat{H}\mathcal{T}^{-1} = \hat{H} & \hat{H} \text{ is invariant under } \mathcal{T}. \\ \mathcal{T}\, i\, \mathcal{T}^{-1} = -i & \mathcal{T} \text{ is anti-unitary.} \end{cases}$$

Choose a basis, $\mathcal{T} = T \cdot K$

$\uparrow$ unitary matrix $\nwarrow$ complex conjugation operator

$$\mathcal{T}^2 = e^{i\phi} I$$

$$\Rightarrow T \underbrace{K T K} = e^{i\phi}$$

$$\Rightarrow T T^* = e^{i\phi}$$

$$\Rightarrow T^* = e^{i\phi} T^\dagger = e^{i\phi} (T^*)^T$$

$$\Rightarrow T^* = e^{i\phi} (T^*)^T = e^{i\phi} \left(e^{i\phi} (T^*)^T\right)^T = e^{2i\phi} T^*$$

$$\Rightarrow e^{2i\phi} = 1$$

$$\Rightarrow \mathcal{T}^2 = T T^* = \pm 1$$

$$T^* = \pm T^\dagger = (\pm T^T)^* \Rightarrow T^T = \pm T$$

$$\begin{cases} \mathcal{T}^2 = +1 \iff T^T = T, \ T \text{ is symmetric} \\ \mathcal{T}^2 = -1 \iff T^T = -T, \ T \text{ is skew-symmetric.} \end{cases}$$

Example. For spin-$\frac{1}{2}$ system, $\mathcal{T}: \vec{S} \to -\vec{S}$, $\vec{S} = \frac{1}{2}\vec{\sigma}$

$$\mathcal{T} = T \cdot K$$

$$\mathcal{T} \frac{\vec{\sigma}}{2} \mathcal{T}^{-1} = -\frac{\vec{\sigma}}{2}$$

$$\Rightarrow \mathcal{T} = i\sigma_y K$$

$$\Rightarrow \mathcal{T}^2 = -1$$

For spin-S system, $\mathcal{T}: \vec{S} \to -\vec{S}$, $\vec{S} = \sum_{j=1}^{2S} \frac{1}{2} \vec{\sigma}_j$

$$\Rightarrow \mathcal{T}^2 = (-1)^{2S} = \begin{cases} 1, & S \text{ integer} \\ -1, & S \text{ half-odd-integer} \end{cases}$$

Kramers doublet:

For $\mathcal{T}$ invariant system with $\mathcal{T}^2 = -1$, $|a\rangle$ and $|b\rangle = \mathcal{T}|a\rangle$ are two different states with the same energy.

(If $\mathcal{T}|a\rangle = e^{i\phi}|a\rangle$, then

$$\mathcal{T}^2|a\rangle = \mathcal{T}(\mathcal{T}|a\rangle) = \mathcal{T} e^{i\phi}|a\rangle = e^{-i\phi}\mathcal{T}|a\rangle = e^{-i\phi}e^{i\phi}|a\rangle = |a\rangle$$

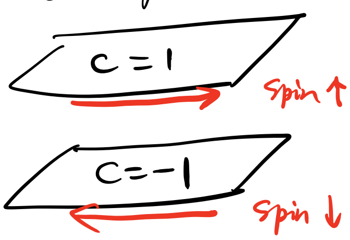
$$\Rightarrow \mathcal{T}^2 = +1$$

• 2D quantum Spin Hall effect.

QHE / Chern insulator breaks $\mathcal{T}$: $\vec{B} \to -\vec{B}$

chiral current: $\vec{j} \to -\vec{j}$

For spin-$\frac{1}{2}$ system,



$$\mathcal{H}_{\vec{k}} = \sin k_x\, \sigma_x + \sin k_y\, \sigma_y + (u + \cos k_x + \cos k_y)\, \sigma_z$$

$$\downarrow$$

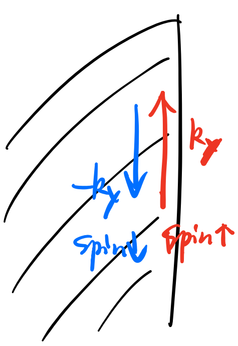
$$\mathcal{H}_{\vec{k}} = \sin k_x \underset{\text{spin}}{s_z} \otimes \underset{\text{band}}{\sigma_x} + s_0 \otimes \left[\sin k_y\, \sigma_y + (u + \cos k_x + \cos k_y)\, \sigma_z\right]$$

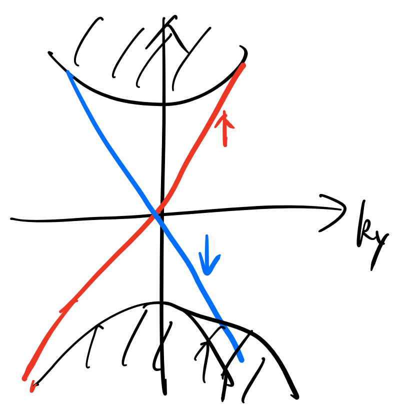
BHZ
||
Bernevig-Hughes-Zhang

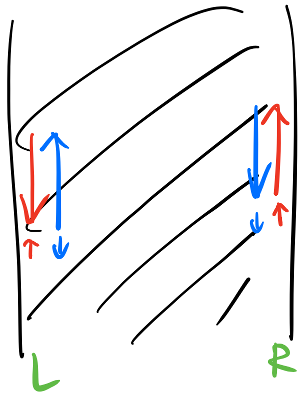
$\{ s_z \otimes \sigma_x,\ s_0 \otimes \sigma_y,\ s_0 \otimes \sigma_z \}$ anticommute with each other.

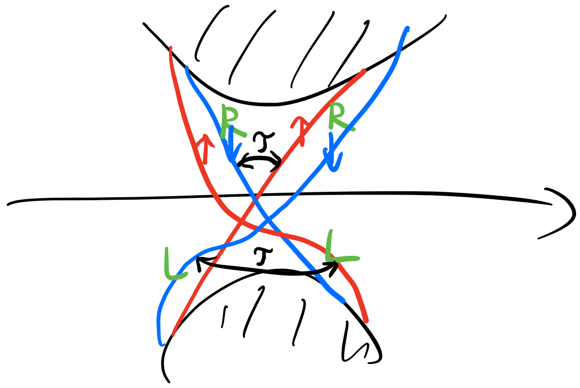
$$= \begin{pmatrix} \sin k_x\, \sigma_x + \sin k_y\, \sigma_y + (u + \cos k_x + \cos k_y)\, \sigma_z & 0 \\ 0 & -\sin k_x\, \sigma_x + \sin k_y\, \sigma_y + (u + \cos k_x + \cos k_y)\, \sigma_z \end{pmatrix}$$

$$\to \begin{pmatrix} c = 1 & \\ & C = -1 \end{pmatrix}$$









6.5. Edge states in different dimensions.

	bulk H (near critical pt)	edge eigenvalue prob.	edge state
1D.	$\mathcal{H}=k_x\sigma_x+m\sigma_z$ $=k_1\sigma^1+m\sigma^3$	$\underline{[-i\sigma_x\partial_x+m(x)\sigma_z]}\psi(x)=0$ $\Leftrightarrow [-\partial_x-m(x)\sigma_y]\psi(x)=0$ $\psi(x)=\varphi(x)\binom{1}{i}$ $\Leftrightarrow [-\partial_x-m(x)]\varphi(x)\binom{1}{i}=0$	$e^{-\lvert x\rvert}\lvert\sigma_y=1\rangle$
2D.	$\mathcal{H}=k_x\sigma_x+k_y\sigma_y+m\sigma_z$ $=k_1\sigma^1+k_2\sigma^2+m\sigma^3$	$[-i\sigma_x\partial_x+k_y\sigma_y+m(x)\sigma_z]\psi$ $\psi(x)=\varphi\cdot\binom{1}{i}$ $=[\underline{-i\sigma_x\partial_x}+k_y+\underline{m(x)\sigma_z}]\psi$ $=k_y\psi$	$e^{-\lvert x\rvert}e^{ik_y y}\lvert\sigma_y=1\rangle$
3D.	$\mathcal{H}=k_1\sigma^{10}+k_2\sigma^{21}+k_3\sigma^{23}+m\sigma^{30}$ 2 bands, spin $\sigma^{ij}:=\sigma^i\otimes\sigma^j$ $(c_1^\dagger, c_2^\dagger)$ $(c_{1\uparrow}^\dagger, c_{1\downarrow}^\dagger, c_{2\uparrow}^\dagger, c_{2\downarrow}^\dagger)$ $c_1\quad c_2\quad c_3\quad c_4$	$[-i\sigma^{10}\partial_x+k_2\sigma^{21}+k_3\sigma^{23}+m(x)\sigma^{30}]\psi$ $\psi=\varphi(x)\binom{1}{i}\chi$ $=[\underline{-i\sigma^{10}\partial_x}+k_2\sigma^{21}+k_3\sigma^{23}\underline{+m(x)\sigma^{30}}]\varphi(x)\binom{1}{i}\chi$ $=(k_2\sigma^{21}+k_3\sigma^{23})\varphi(x)\binom{1}{i}\chi$ $=(k_2\sigma^1+k_3\sigma^3)\varphi(x)\binom{1}{i}\chi$ $=\varphi(x)\cdot\binom{1}{i}\cdot\underbrace{(k_2\sigma^1+k_3\sigma^3)\chi}$	$e^{-\lvert x\rvert}\lvert\sigma_y=1\rangle\otimes\lvert\chi\rangle$ $\mathcal{H}=k_2\sigma^1+k_3\sigma^3$

relations of TI in different dims.