Part II. Topological insulators and superconductors

(= fermionic $\underbrace{\text{symmetry-protected topological}}_{\text{SPT}}$ phases w/o interactions)

	noninteracting	interacting
bosonic	/	bSPT
fermionic	TI/TSC	fSPT

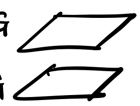
G-SPT : states without anyons, but still can NOT be deformed into trivial product state preserving symmetry G.

Topological phase ⟶ Abelian monoid under stacking.

$\downarrow$ ∃ inverse

G-SPT ($\in$ invertible phases) ⟶ Abelian group under stacking.

classified by $\mathbb{Z}$, $\mathbb{Z}_n$ or direct sum $\oplus$ of them.

Symmetry action on stacked system : G  G

coproduct : $G \to G \times G$

$g \mapsto g \otimes g$

$U(g) \mapsto U_1(g) \otimes U_2(g)$

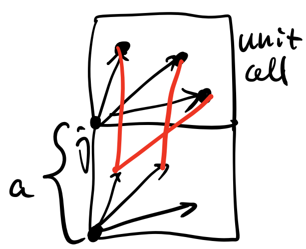
5. Integer quantum Hall effect and Chern insulators.

Noninteracting TI with symmetry group $U(1)_c = U(1)_f$

5.1. Band theory.

free fermions hopping on lattice

$$H = \sum_{j,\delta,m,n} t_\delta^{mn} \, c^\dagger_{j+\delta,m} \, c_{j,n} + h.c.$$



Fourier transformation $c_{j,n} = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{j}}\, c_{kn}$

linear system size

$$\begin{cases} c_{j+L,n} = c_{j,n} \ (\text{finite system}) \Rightarrow e^{ik_j \cdot L} = 1 \Rightarrow k \in \frac{2\pi}{L}\mathbb{Z} \ (\text{momentum quantization}) \\ j \in a\mathbb{Z} \ (\text{discrete lattice}) \Rightarrow k + \frac{2\pi}{a} \sim k \Rightarrow k \in [0, \frac{2\pi}{a}) \\ \qquad (\text{periodicity of momentum}) \end{cases}$$

$$\Rightarrow k = 0, \frac{2\pi}{L}, \frac{2\pi}{L}\times 2, \cdots, \frac{2\pi}{L}\left(\frac{L}{a}-1\right).$$

Brillouin Zone $= T^d$ ← space dim of the lattice.

$$H = \sum_{j\delta mn} t_\delta^{mn} \frac{1}{N} \sum_{\vec{k},\vec{k}' \in BZ} e^{-i\vec{k}\cdot(\vec{j}+\vec{\delta}) + i\vec{k}'\cdot\vec{j}}\, c^\dagger_{\vec{k},m}\, c_{\vec{k}',n} + h.c.$$

$$= \sum_{\vec{k},\vec{k}' \in BZ} \sum_{mn\delta} \underbrace{\left[\frac{1}{N}\sum_{j} e^{i(\vec{k}'-\vec{k})\cdot\vec{j}}\right]}_{\delta_{\vec{k}\vec{k}'}} t_\delta^{mn} e^{-i\vec{k}\cdot\vec{\delta}}\, c^\dagger_{\vec{k},m}\, c_{\vec{k}',n} + h.c.$$

$$= \sum_{\vec{k}\in BZ} \sum_{m,n} \underbrace{\left(\sum_{\delta} t_\delta^{mn} e^{-i\vec{k}\cdot\vec{\delta}}\right)}_{=: t_{\vec{k}}^{m,n}} c^\dagger_{k,m}\, c_{k,n} + h.c.$$

$$= \sum_{\vec{k}\in BZ} \left(c^\dagger_{k,1}, \cdots, c^\dagger_{k,M}\right) \mathcal{H}_{\vec{k}} \begin{pmatrix} c_{k,1} \\ \vdots \\ c_{k,M} \end{pmatrix}$$

$$(\mathcal{H}_{\vec{k}})_{m,n} = \sum_{\delta}\left(t_\delta^{mn} e^{-i\vec{k}\cdot\vec{\delta}} + t_\delta^{nm*} e^{i\vec{k}\cdot\vec{\delta}}\right)$$

$$\mathcal{H}: \quad BZ = T^d \longrightarrow \mathrm{Mat}_M(\mathbb{C})$$
$$\qquad \vec{k} \longmapsto \mathcal{H}_{\vec{k}}$$

Adding symmetries to system $\Leftrightarrow$ adding constraints on target manifold.

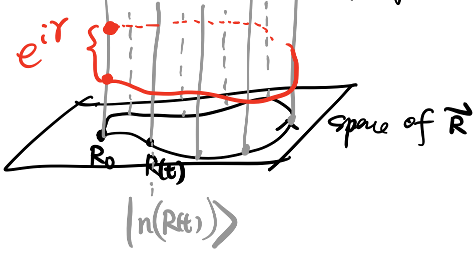
Classification of TI $\Leftrightarrow$ finding homotopy classes of the map $\mathcal{H}$.

5.2. Berry phase.

Assume a system depends on parameters $\vec{R} = (R_1, \cdots, R_N)$

$$H(\vec{R})\,|n(\vec{R})\rangle = E_n(\vec{R})\,|n(\vec{R})\rangle$$

$|n(\vec{R})\rangle \sim e^{i\theta}|n(\vec{R})\rangle$ are the same physical state.



If the parameter $\vec{R}(t)$ is slowly changed with time t:

$$i\frac{d}{dt}|\psi(t)\rangle = H(\vec{R}(t))\,|\psi(t)\rangle.$$

Assume $|\psi(t)\rangle = e^{i\gamma_n(t)}\, e^{-i\int_0^t E_n(R(t'))\,dt'}\,|n(\vec{R}(t))\rangle$, then

$$i\frac{d}{dt}|\psi(t)\rangle = -\frac{d\gamma_n(t)}{dt}|\psi(t)\rangle + \cancel{E_n(R(t))\,|\psi(t)\rangle} + e^{i\gamma_n(t)}e^{-i\int_0^t E_n(R(t'))dt'}\, i\frac{d}{dt}|n(\vec{R}(t))\rangle$$

$$= e^{i\gamma_n(t)}e^{-i\int_0^t E_n(R(t'))dt'}\underbrace{H(\vec{R}(t))|n(\vec{R}(t))\rangle}_{\cancel{E_n(R(t))|n(\vec{R}(t))\rangle}}$$

$$\Rightarrow -\frac{d\gamma_n(t)}{dt}|\psi(t)\rangle + e^{i\gamma_n(t)}e^{-i\int_0^t E_n(R(t'))dt'}\, i\frac{d}{dt}|n(\vec{R}(t))\rangle = 0$$

$$\Rightarrow -\frac{d\gamma_n(t)}{dt}|n(R(t))\rangle + i\frac{d}{dt}|n(\vec{R}(t))\rangle = 0$$

$$\Rightarrow \frac{d\gamma_n(t)}{dt} = i\langle n(R(t))|\frac{d}{dt}|n(R(t))\rangle$$

$$= i\frac{d\vec{R}}{dt}\cdot\langle n(R(t))|\nabla_{\vec{R}}|n(R(t))\rangle$$

$$\Rightarrow \gamma_n(L) = \int_0^t dt\,\frac{d\gamma_n(t)}{dt} = \int_0^t i\langle n(\vec{R})|\nabla_{\vec{R}}|n(\vec{R})\rangle\cdot\frac{d\vec{R}}{dt}\cdot dt$$

$$= \oint_R i\langle n(R)|\nabla_{\vec{R}}|n(R)\rangle\cdot d\vec{R} \rightarrow \text{Berry phase}$$

Berry connection $\vec{A}_n(\vec{R}) := i\langle n(\vec{R})|\nabla_{\vec{R}}|n(\vec{R})\rangle$

Gauge transformation: $|n'(\vec{R})\rangle := e^{i\alpha(\vec{R})}|n(\vec{R})\rangle$

$$\Rightarrow A_n'(\vec{R}) := i\langle n'(R)|\nabla_{\vec{R}}|n'(R)\rangle$$

$$= i\langle n(R)|\left(i e^{i\alpha(\vec{R})}\nabla_{\vec{R}}\alpha(\vec{R}) + e^{i\alpha(\vec{R})}\nabla_{\vec{R}}\right)|n(\vec{R})\rangle$$

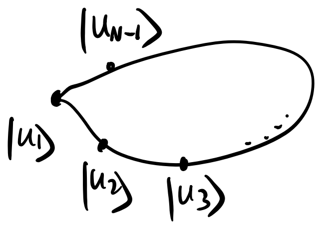
$$= -\langle n(R)|n(\vec{R})\rangle\,\nabla_{\vec{R}}\alpha(\vec{R}) + i\langle n(R)|\nabla_{\vec{R}}|n(\vec{R})\rangle$$

$$= A_n(\vec{R}) - \nabla_{\vec{R}}\, \alpha(\vec{R})$$

$$\Rightarrow \gamma_n'(L) = \oint_L A_n'(\vec{R}) = \oint_L \left[A_n(\vec{R}) - \nabla_{\vec{R}}\, \alpha(\vec{R})\right] = \gamma_n(L)$$

Berry phase $\gamma_n(L) := \oint_L \vec{A}_n(\vec{R}) \cdot d\vec{R}$ is gauge invariant for closed loop L in the parameter space of $\vec{R}$.

- discrete $\rightarrow$ continuous



$$|u_j\rangle \rightarrow |u_{j+1}\rangle$$

$$\langle u_j | u_{j+1}\rangle = |\langle u_j | u_{j+1}\rangle| \cdot e^{i \arg \langle u_j | u_{j+1}\rangle}$$

phase difference $\arg \langle u_j | u_{j+1}\rangle = \operatorname{Im} \ln \langle u_j | u_{j+1}\rangle$

Total phase difference $\gamma = \sum_j \operatorname{Im} \ln \langle u_j | u_{j+1}\rangle = \operatorname{Im} \ln\left(\langle u_0|u_1\rangle\langle u_1|u_2\rangle \cdots \cdots \langle u_{N-1}|u_0\rangle\right)$

$$|u_{\vec{R}}\rangle \rightarrow |u_{\vec{R}+d\vec{R}}\rangle = |u_{\vec{R}}\rangle + d\vec{R} \cdot \nabla_{\vec{R}} |u_{\vec{R}}\rangle + \cdots$$

$$\Rightarrow \langle u_{\vec{R}} | u_{\vec{R}+d\vec{R}}\rangle \approx 1 + d\vec{R}\, \langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle$$

$$\Rightarrow \ln \langle u_{\vec{R}} | u_{\vec{R}+d\vec{R}}\rangle \approx \ln\left[1 + d\vec{R}\, \langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle\right]$$

$$\approx d\vec{R} \cdot \langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle$$

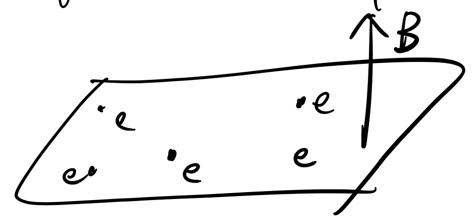
$$\Rightarrow \gamma = \operatorname{Im} \oint d\vec{R} \cdot \underline{\langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle} = \oint d\vec{R} \cdot \underbrace{i \langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle}_{\vec{A}_{\vec{R}}}$$

$$\boxed{\begin{aligned} \left(\langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle\right)^* &= \left(\langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle\right)^+ \\ &= \left(\nabla_{\vec{R}} |u_{\vec{R}}\rangle\right)^+ |u_{\vec{R}}\rangle = \left(\langle u_{\vec{R}}| \nabla_{\vec{R}}\right) |u_{\vec{R}}\rangle \\ &= -\langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle \\ \Rightarrow \langle u_{\vec{R}} | \nabla_{\vec{R}} | u_{\vec{R}}\rangle &\text{ is purely imaginary.} \end{aligned}}$$

Non-Abelian generalization:

$$A_{mn}(\vec{R}) := i \langle m(\vec{R}) | \nabla_{\vec{R}} | n(\vec{R}) \rangle$$

5.3. Integer quantum Hall effect.



$$H = \frac{1}{2m}(-i\vec{\nabla} + e\vec{A})^2$$

$$\vec{A} \Rightarrow \vec{B} = B\hat{z}$$

Landau gauge : $\vec{A} = Bx\,\hat{y} = (0, Bx, 0)$

$$\Rightarrow B_z := \partial_x A_y - \partial_y A_x = \partial_x(B \cdot x) = B$$

$$\Rightarrow \vec{B} = B\hat{z} = (0, 0, B)$$

$$H = \frac{1}{2m}(-i\vec{\nabla} + e\vec{A}) = \frac{1}{2m}\left[-\partial_x^2 + (-i\partial_y + eBx)^2\right]$$

$[H, p_y] = 0 \Rightarrow H$ is invariant under y translation.

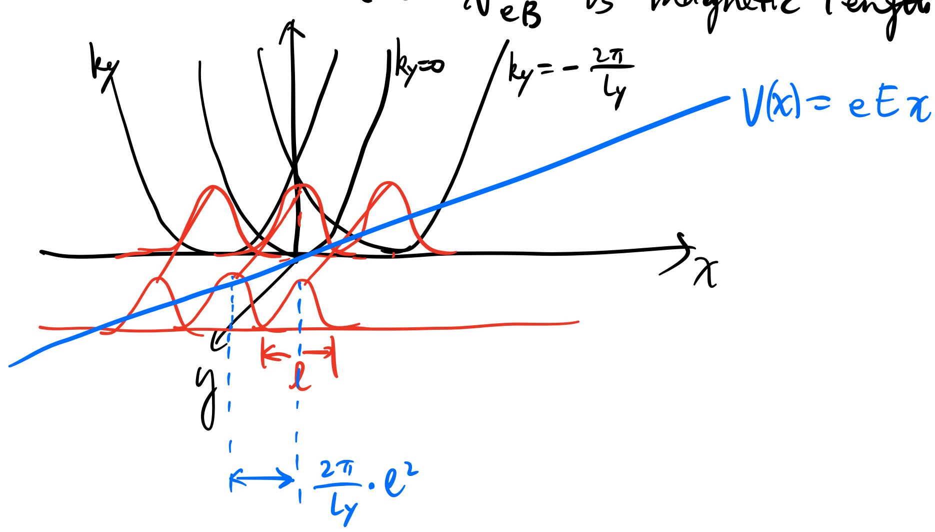
$$\psi(x,y) = e^{ik_y y}\,\psi_{k_y}(x)$$

$$H = \sum_{k_y} \frac{1}{2m}\left[-\partial_x^2 + (k_y + eBx)^2\right]$$

$\rightarrow$ a family of 1D harmonic oscillator parametrized by k_y.

$$H = \sum_{k_y}\left[-\frac{1}{2m}\hbar^2\partial_x^2 + \frac{1}{2}m\omega^2(x + k_y\ell^2)^2\right]$$

where $\begin{cases} \omega = \frac{eB}{m} & \text{is frequency of HO.} \\ \ell = \sqrt{\frac{\hbar}{eB}} & \text{is magnetic length.} \end{cases}$



$$\Rightarrow \begin{cases} \epsilon_n = \hbar\omega\left(n+\frac{1}{2}\right),\ n\in\mathbb{Z} \quad \text{for } \forall k_y \\ \text{Lowest Landau level } (n=0) \text{ (LLL)}: \quad \text{Gaussian wavefunction} \\ \qquad \text{center } X_{k_y} = -k_y \ell^2, \ \text{size } \ell \\ \qquad \psi(x,y) \propto e^{ik_y\cdot y}\, e^{-\frac{1}{2\ell^2}(x-X_{k_y})^2} \\ \text{degeneracy for } \epsilon_n\ (\forall n): \quad \dfrac{L_x}{\frac{2\pi}{L_y}\cdot \ell^2} = \dfrac{L_x L_y}{2\pi \ell^2} = \dfrac{L_x L_y B}{2\pi\hbar/e} = \dfrac{B L_x L_y}{\Phi_0} \\ \qquad = \dfrac{\Phi}{\Phi_0} \end{cases}$$

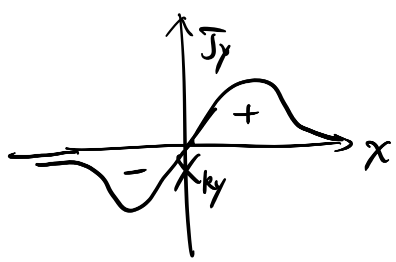
$\Phi_0 = \frac{h}{e}$ is the flux quantum.

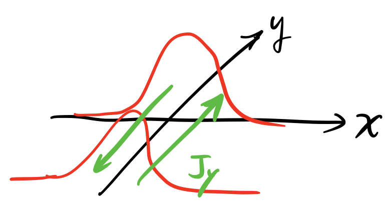
- Current.

$$\hat{J}_y = \frac{-e}{m}\left(\hat{p}_y - eA_y\right)$$

$$\langle LLL|\hat{J}_y|LLL\rangle \propto \int dx\, e^{-\frac{1}{2\ell^2}(x-X_{k_y})^2}(\hbar k_y + eBx)\, e^{-\frac{1}{2\ell^2}(x-X_{k_y})^2}$$

$$\propto \int dx\, e^{-\frac{1}{\ell^2}(x-X_{k_y})^2}(x-X_{k_y}) = 0$$





Add an electric field in X direction $V(x) = eEx$.

$$H = \sum_{k_y} -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2(x+k_y\ell^2)^2 + eEx$$

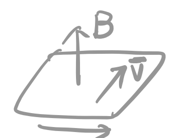
$$= \sum_{k_y} -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2\left(x+k_y\ell^2+\frac{eE}{m\omega^2}\right)^2 - eEX'_{k_y} + \frac{1}{2}m\bar{v}^2$$

$-k_y\ell^2 - \frac{eE}{m\omega^2}$ is the new center

$\bar{v} = -\frac{E}{B}$

$$\Rightarrow \epsilon_{k_y} = \frac{1}{2}\hbar\omega_c - eEX'_{k_y} + \frac{1}{2}m\bar{v}^2$$

$$\Rightarrow \text{group velocity} \quad V_y^{group} := \frac{\partial \epsilon_{k_y}}{\partial(\hbar k_y)} = \frac{eE}{\hbar}\frac{\partial X'_{k_y}}{\partial k_y} = -\frac{E}{B} = \bar{v}$$

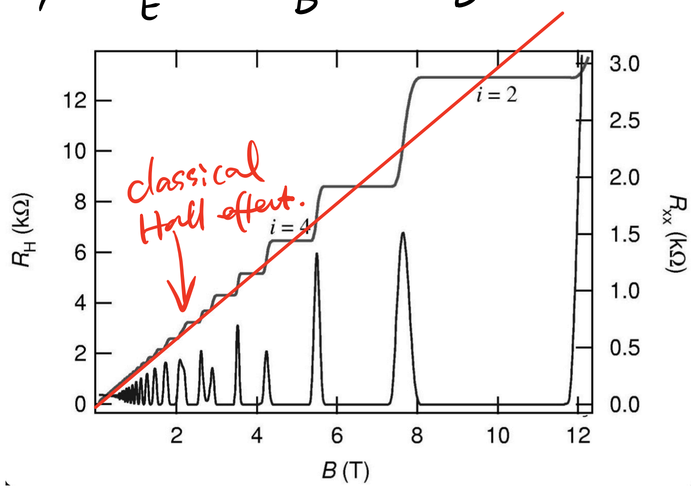


$qE = qvB$

$E \quad \Rightarrow V = \frac{E}{B}$

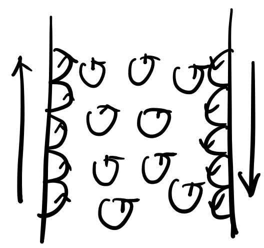
$$\Rightarrow \langle J_y \rangle = \rho(-e)\bar{v} = \frac{\rho e E}{B}$$

$$\Rightarrow \sigma_{xy} = \frac{j_y}{E} = \frac{\rho e}{B} \propto \frac{1}{B} \longrightarrow \text{classical Hall effect.}$$



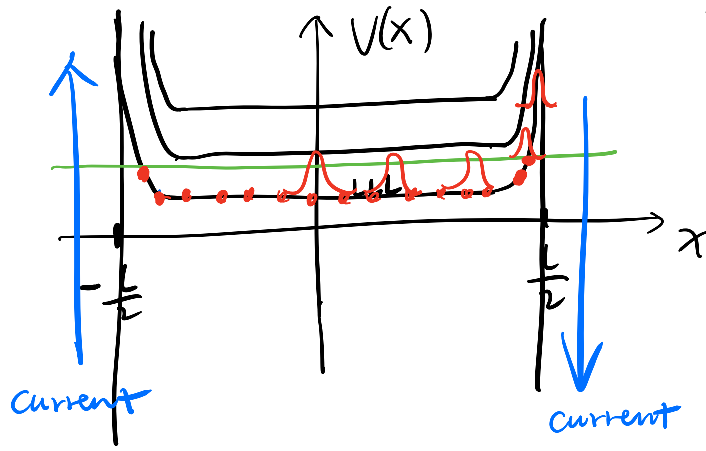
- Edge state.

classical picture :



Add a slowly-changing potential $V(x)$ to confine the electrons.

$V'(x) \cdot \ell \ll w$.



$$V_y^{group} = \frac{\partial \epsilon_k}{\hbar \partial k_y} = \frac{1}{\hbar}\frac{\partial \epsilon_k}{\partial X_k}\frac{\partial X_k}{\partial k} = -\frac{\ell^2}{\hbar}\frac{\partial \epsilon_k}{\partial X_k} = \begin{cases} <0, & \text{right-edge} \\ >0, & \text{left-edge} \end{cases}$$

$$I_y = -e\int_{-\infty}^{+\infty}\frac{dk_y}{2\pi}\, V_y^{group}\, n_{k_y}$$

occupation number of k_y'th mode.

$$= -\frac{e}{h}\int_{\mu_L}^{\mu_R} d\epsilon = -\frac{e}{\hbar}(\mu_R - \mu_L) = -\frac{e}{h} eV$$

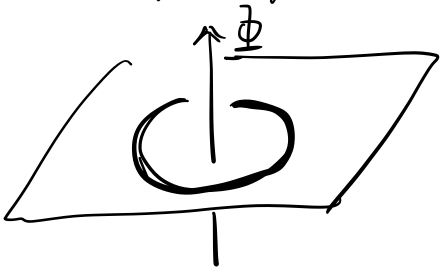
chemical potential

$$\Rightarrow \sigma_{xy} = -\frac{e^2}{h}$$

in general $\sigma_{xy} = -\nu \frac{e^2}{h}$, $\nu \in \mathbb{Z}$.

5.4. Laughlin argument

- flux quantization.



$$H = \frac{1}{2m}\left(-i\hbar \partial_x - eA\right)^2 = \frac{1}{2m}\left(-i\hbar D_x\right)^2$$

$$\psi(x+2\pi R) = \psi(x)$$

gauge transf: $\begin{cases} A' = A + \frac{\hbar}{eR} n \\ \psi'(x) = \psi(x)\, e^{i\frac{n}{R}x} \end{cases}$ $(n \in \mathbb{Z})$

$$\Rightarrow \begin{cases} D_x' \psi'(x) = D_x \psi(x) \\ \psi'(x+2\pi R) = \psi'(x)\, e^{i2\pi n} = \psi'(x) \end{cases}$$

$\Rightarrow$ H and H' are gauge equivalent.

$$\Phi' = A' \cdot 2\pi R = \left(A + \frac{\hbar}{eR} n\right) 2\pi R = \Phi + n\frac{h}{e}$$

flux quantum $\Phi_0 = \frac{h}{e}$

Systems with Φ and $\Phi + n\Phi_0$ can be related by gauge transformation.
$H(\Phi)$ $H(\Phi + n\Phi_0)$

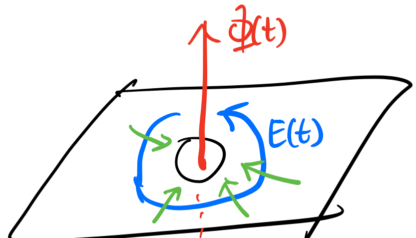
Path integral picture:

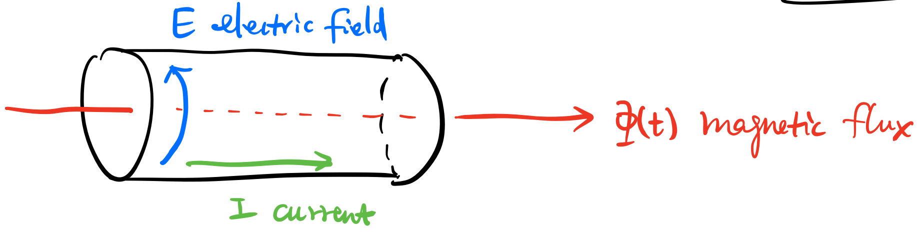
$(\text{phase}) = e^{i\frac{e}{\hbar}\oint_L A_\mu dx^\mu}$ for an electron along loop L.

If $\Phi \to \Phi + n\frac{h}{e}$, $(\text{phase}) \to (\text{phase}) \cdot e^{i2\pi n} = (\text{phase})$

$\Rightarrow$ Same partition function.

- Laughlin argument.





$$\oint E\, dl = \partial_t \Phi \Rightarrow E(t) = \frac{1}{2\pi R} \partial_t \Phi(t)$$

Consider $\Phi(t=0)=0 \longrightarrow \Phi(t=T) = \Phi_0$, then $H(\Phi=0) \sim H(\Phi=\Phi_0)$

$$\Delta Q = \nu e \quad (\nu \in \mathbb{Z})$$

$$\Rightarrow I = \frac{\Delta Q}{T} = \frac{\nu e}{T}$$

$$\left.\begin{aligned} \Rightarrow j &= \frac{I}{2\pi R} = \frac{\nu e}{2\pi R T} \\ E &= \frac{\Phi_0}{2\pi R T} \end{aligned}\right\} \Rightarrow \sigma_{xy} = \frac{j}{E} = \frac{e\nu}{\Phi_0} = \nu \frac{e^2}{h} \quad (\nu \in \mathbb{Z})$$

is quantized.

$$\left.\begin{aligned} &H(\Phi + \Phi_0) \sim H(\Phi) \\ &\text{state goes back to itself}, \Delta Q \in e\mathbb{Z} \end{aligned}\right\} \Rightarrow \sigma_{xy} \in \frac{e^2}{h}\mathbb{Z}$$

5.6. Thouless-Kohmoto-Nightingale-den Nijs (TKNN) number and Chern insulator.

$$\begin{cases} \text{Band theory} & \mathcal{H}_{\vec{k}},\ |\psi_{\vec{k}}\rangle \text{ for } \vec{k} \in T^2 \\ \text{Berry phase} & \gamma(L) = \oint_L d\vec{R} \cdot \vec{A},\quad \vec{A} = i \langle \psi | \nabla_{\vec{R}} | \psi \rangle \end{cases}$$

$\Rightarrow$ Berry phase for a free fermion band.

Berry connection: $A_\alpha := i \langle \psi_{\vec{k}} | \partial_{k_\alpha} | \psi_{\vec{k}} \rangle$

Berry curvature: $F_{xy} := \partial_x A_y - \partial_y A_x$

TKNN number = Chern number : $C := \frac{1}{2\pi} \int_{T^2} d^2k\, F_{xy}$

Kubo formula $\Rightarrow$ $\sigma_{xy} = \frac{e^2}{h}\, C$

physical quantity $\qquad$ topological invariant.

Example of 2-band model.

$$\mathcal{H} : T^2 \to \mathrm{Mat}_2(\mathbb{C})$$

$$\vec{k} \mapsto \mathcal{H}_{\vec{k}} = \vec{E}(\vec{k})\cdot\vec{\sigma} \quad + \epsilon(\vec{k})\cdot I_{2\times 2}$$

$$= |\vec{E}(\vec{k})| \underbrace{\frac{\vec{E}(\vec{k})}{|\vec{E}(\vec{k})|}}_{=\hat{n}(\vec{k})} \cdot \vec{\sigma}$$

$$= E(k)\ \hat{n}(\vec{k})\cdot\vec{\sigma}$$

$$= E(k)\left[\, n_x(\vec{k})\,\sigma^x + n_y(\vec{k})\,\sigma^y + n_z(\vec{k})\,\sigma^z \right]$$

with $|\hat{n}(\vec{k})| = 1$

eigenvalues of $\mathcal{H}_{\vec{k}}$ is $\pm E(\vec{k})$.

$$\hat{n} : T^2 \to S^2$$

$$\vec{k} \mapsto \hat{n}(\vec{k}) = \big(\sin\theta(\vec{k})\cos\varphi(\vec{k}),\ \sin\theta(\vec{k})\sin\varphi(\vec{k}),\ \cos\theta(\vec{k})\big) \in S^2$$

$$|\psi_+\rangle = \begin{pmatrix} \cos\frac{\theta(\vec{k})}{2} \\ e^{i\varphi(\vec{k})}\sin\frac{\theta(\vec{k})}{2} \end{pmatrix}$$

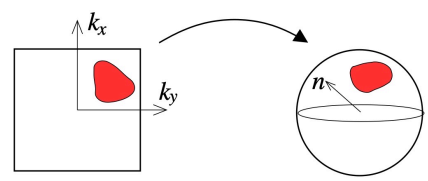
$$\Rightarrow C = \frac{1}{2\pi}\int_{T^2} d^2k\, F_{xy}$$

$$= \cdots$$

$$= \frac{1}{4\pi}\int_{T^2} d^2k\ \hat{n}(\vec{k})\left[\frac{\partial \hat{n}(\vec{k})}{\partial k_x} - \frac{\partial \hat{n}(\vec{k})}{\partial k_y}\right]$$

$= \frac{1}{4\pi}\int_{T^2} \hat{n}^*(\omega)$ $\quad$ where ω is the volumn form of S^2

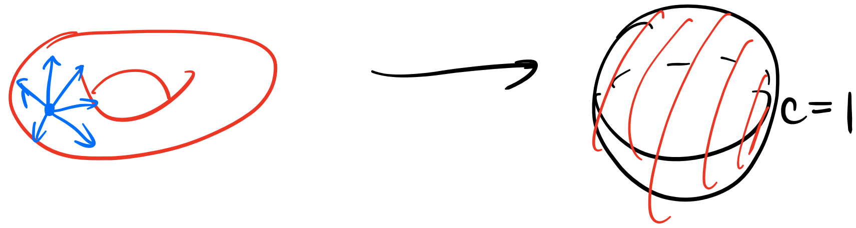
$=$ mapping degree $(\hat{n})$ $=$ # of times T^2 wrap around S^2.



Brillouin zone $\qquad$ Bloch sphere

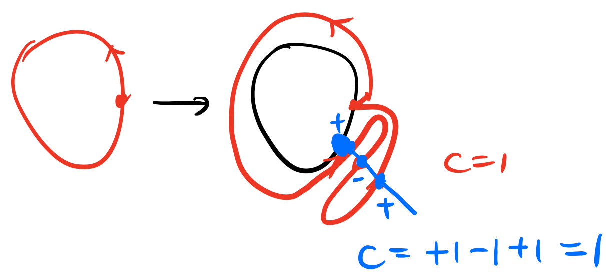
Examples: T^2 in $\mathbb{R}^3$, given $\vec{x}_0 \in \mathbb{R}^3 \setminus T^2$

define $\hat{n}(\vec{k}) := \dfrac{\vec{k} - \vec{x}_0}{|\vec{k} - \vec{x}_0|}$



$$\text{mapping degree}(\hat{n}) = \sum_{\vec{k} \in \hat{n}^{-1}(n_0)} \operatorname{sgn} \det \underbrace{D}_{\text{Jacobi matrix at } \vec{k}} \hat{n}(\vec{k})$$

1D case : $\hat{n} : S^1 \to S^1$



Example. Haldane honeycomb model.