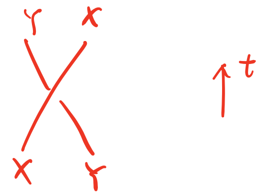
4. Modular tensor categories, Drinfeld center and anyon models.

4.1. Braided monoidal categories

Def. A <u>braided monoidal cat.</u> consists of

- a monoidal cat. $\mathcal{C}$
- a natural isomorphism call braiding that assigns to every pair of objects $X, Y \in \mathcal{C}$ an iso.

$$b_{X,Y} : X \otimes Y \to Y \otimes X$$

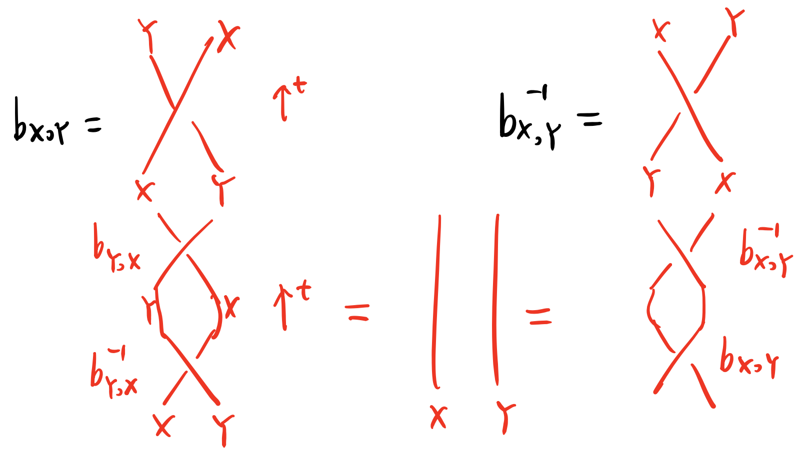


such that the hexagon eq hold:

$$\begin{array}{ccccc}
X\otimes(Y\otimes Z) & \xrightarrow{a^{-1}_{X,Y,Z}} & (X\otimes Y)\otimes Z & \xrightarrow{b_{X,Y}\otimes id_Z} & (Y\otimes X)\otimes Z \\
\downarrow b_{X,Y\otimes Z} & & \circlearrowleft & & \downarrow a_{Y,X,Z} \\
(Y\otimes Z)\otimes X & \xleftarrow{a^{-1}_{Y,Z,X}} & Y\otimes(Z\otimes X) & \xleftarrow{id_Y\otimes b_{X,Z}} & Y\otimes(X\otimes Z)
\end{array}$$

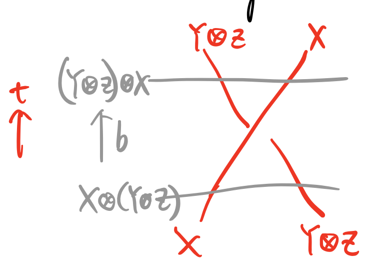
$$\begin{array}{ccccc}
(X\otimes Y)\otimes Z & \xrightarrow{a_{X,Y,Z}} & X\otimes(Y\otimes Z) & \xrightarrow{id_X\otimes b_{Y,Z}} & X\otimes(Z\otimes Y) \\
\downarrow b_{X\otimes Y,Z} & & \circlearrowright & & \downarrow a_{X,Z,Y} \\
Z\otimes(X\otimes Y) & \xleftarrow{a_{Z,X,Y}} & (Z\otimes X)\otimes Y & \xleftarrow{b_{X,Z}\otimes id_Y} & (X\otimes Z)\otimes Y
\end{array}$$

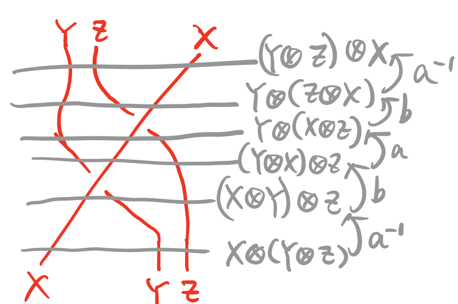
Rem. 1.



$$b_{Y,X} \circ b^{-1}_{Y,X} = id_{X\otimes Y} = b^{-1}_{X,Y} \circ b_{X,Y}$$

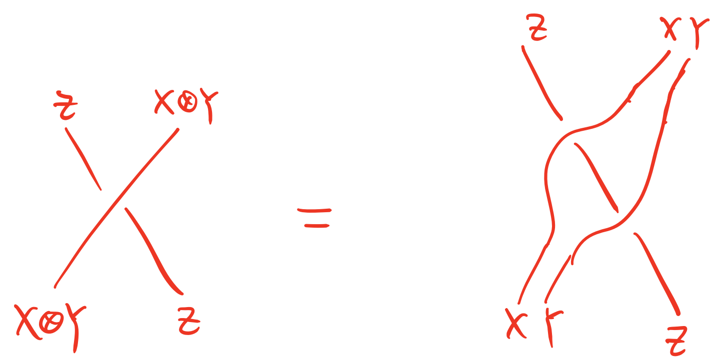
2. hexagon eq. $\Longleftrightarrow$





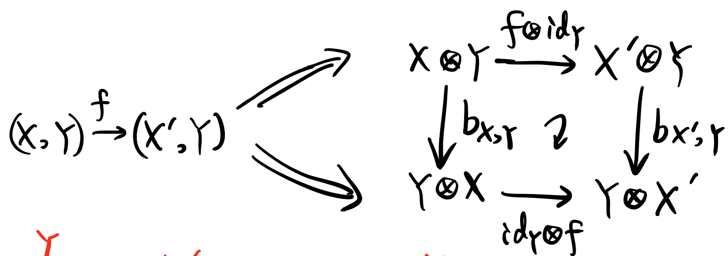
$$b = a^{-1} b a b a^{-1}$$

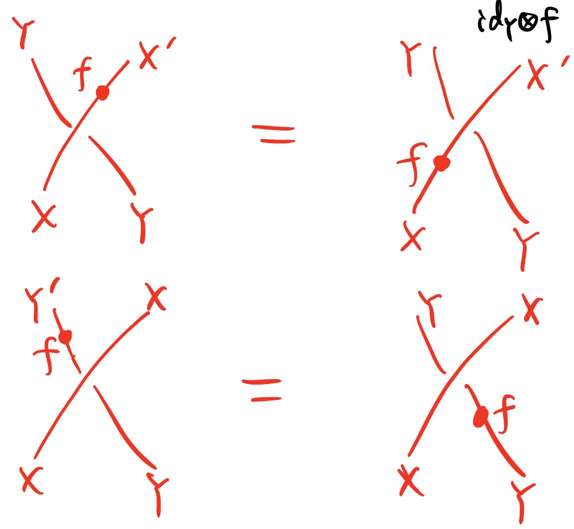
$$\Leftrightarrow aba = bab$$



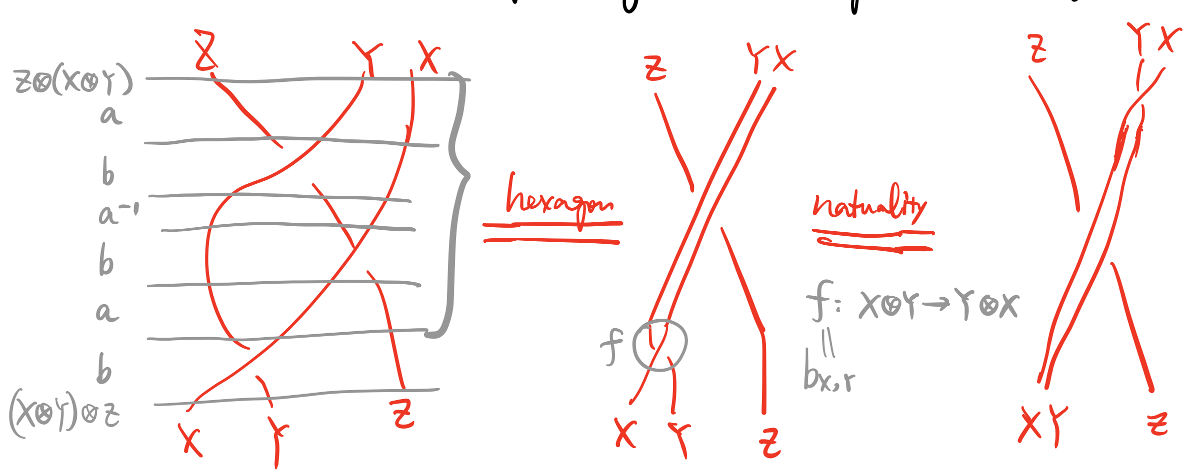
3. naturality of braiding:

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

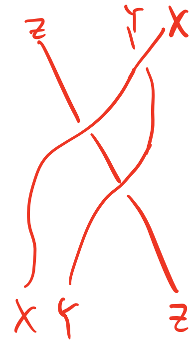




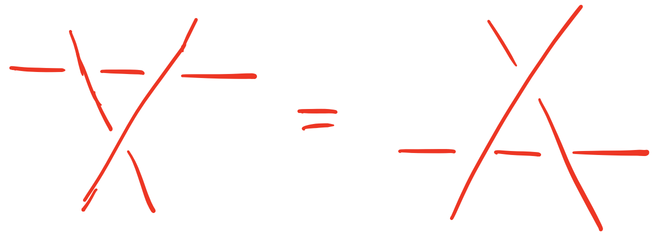
4. naturality + hexagon eq. ⇒ Yang-Baxter eq.



hexagon

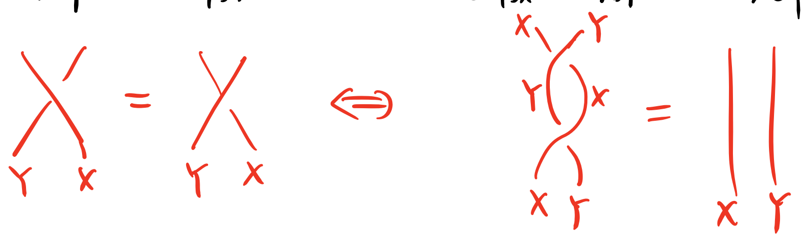


Reidemeister move III for knot ⟿ Yang-Baxter eq.
(geometry) (algebra)



5. $\mathcal{C}$ is called <u>symmetric monoidal category</u> if

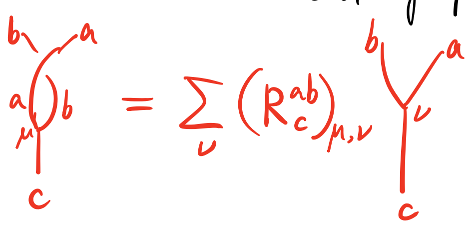
$$b_{X,Y}^{-1} = b_{Y,X} \iff b_{Y,X} \circ b_{X,Y} = id_{X\otimes Y}$$



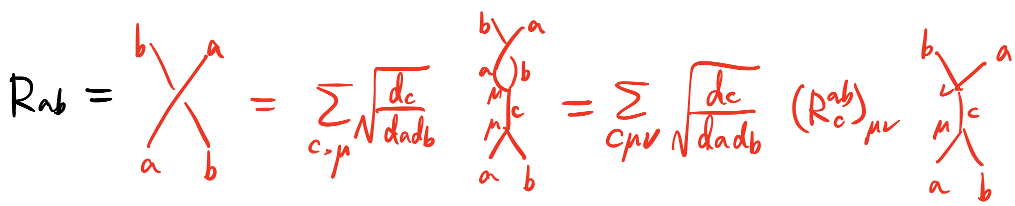
6. String diagram.

Split $a \otimes b = \sum_c N_c^{ab}\, c$ in the simple obj. basis.

braiding ⟶ a trivalent graph in 3D.

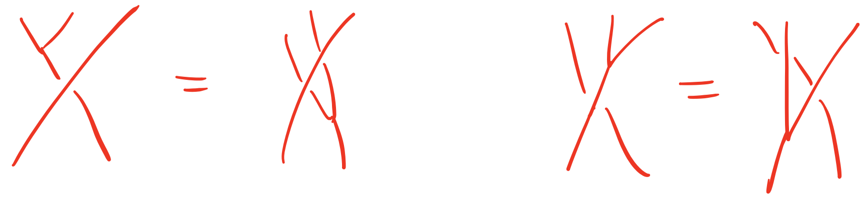


R_c^{ab} is assumed to be unitary.

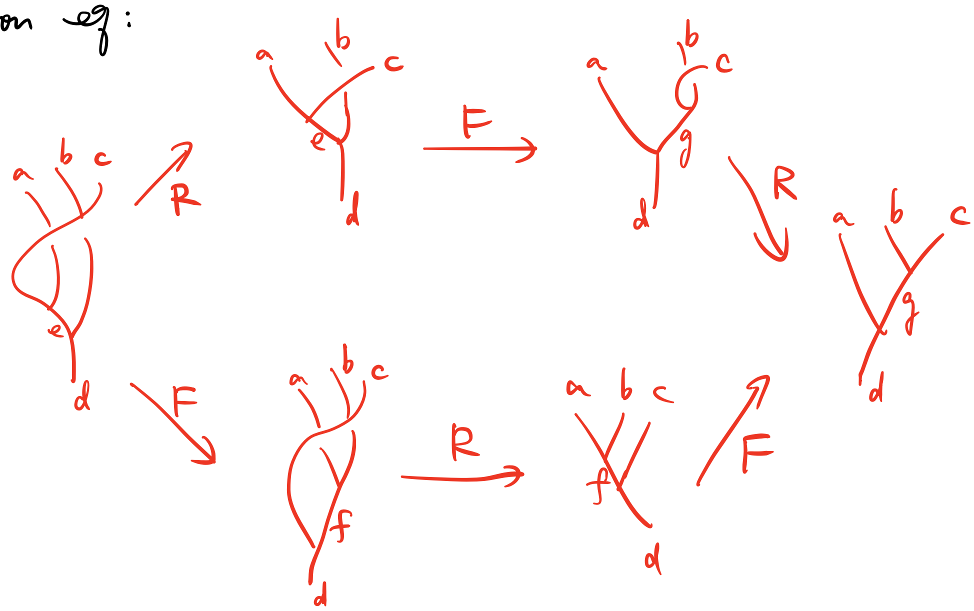


(3D diagram) $\longrightarrow$ (2D diagram)

naturality :



hexagon eq :



$$\sum (R^{\cdot\cdot}_{\cdot})_{\cdot\cdot} (F^{\cdot\cdot\cdot}_{\cdot})_{\cdot\cdot} (R^{\cdot\cdot}_{\cdot})_{\cdot\cdot} = \sum F R F$$

$\forall$ 3D trivalent graph $\xrightarrow{R, F}$ 2D trivalent graph $\xrightarrow{F}$ ϕ

7. modular data.

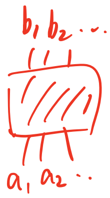
$\theta_a = \frac{1}{d_a}$  $= \sum_{c\mu} \frac{d_c}{d_a} (R^{aa}_c)_{\mu\mu}$

$$\begin{cases} T_{ab} = \theta_a \delta_{a,b} \\ S_{ab} := \frac{1}{D} \ {}_a\text{(link)}_b = \frac{1}{D} \sum_c N^c_{ab} \frac{\theta_c}{\theta_a \theta_b} d_c \end{cases}$$

$\mathcal{C}$ is called <u>modular</u> if S, T are non-singular.

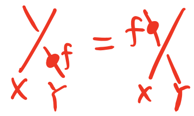
S, T generat rep of $PSL_2(\mathbb{Z})$.

8. 2+1D anyon models are described by a unitary modular tensor category (UMTC)

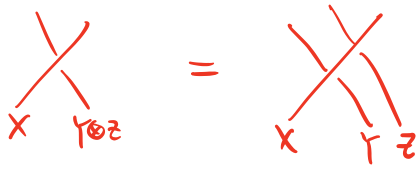
anyon	a	object
anti-particle	a^*	dual object
vacuum	1	identy obj.
fusion of anyons	$a \otimes b$	tensor product
anyon braiding	$R_{a,b}$	braiding
anyon worldline in 2+1 D		string diagram in 3D
transition amplitute		linear map: $a_1 \otimes a_2 \otimes \cdots \longrightarrow b_1 \otimes b_2 \otimes \cdots$
partition function	closed 3D diagram	linear map: $1 \longrightarrow 1$

4.2. Drinfeld center construction.

Let $\mathcal{C}$ be a monoidal category, a <u>half braiding</u> β_X for $X \in \mathcal{C}$ is a family $\{ \beta_X(Y) \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Y \otimes X) \mid Y \in \mathcal{C} \}$ of iso. natural w.r.t. Y,  satisfying $\beta_X(1) = \mathrm{id}_X$ and



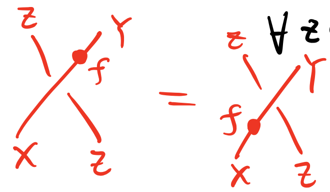
$$\beta_X(Y \otimes Z) = [\mathrm{id}_Y \otimes \beta_X(Z)] \circ [\beta_X(Y) \otimes \mathrm{id}_Z]$$



The <u>Drinfeld center $Z(\mathcal{C})$</u> of $\mathcal{C}$ has obj. (X, β_X), where $X \in \mathcal{C}$ and β_X is a half braiding for X.

The morphisms are

$$\mathrm{Hom}_{Z(\mathcal{C})}((X,\beta_X),(Y,\beta_Y)) := \{ f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \mid [\mathrm{id}_Z \otimes f] \circ \beta_X(Z) = \beta_Y(Z) \circ [f \otimes \mathrm{id}_Z],\ \forall Z \in \mathcal{C} \}$$

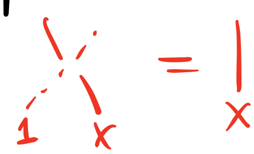


The tensor product in $Z(\mathcal{C})$ is given by

$$(X, \beta_X) \otimes (Y, \beta_Y) = (X \otimes Y, \beta_{X \otimes Y})$$

where $\beta_{X \otimes Y}(Z) := [\beta_X(Z) \otimes \mathrm{id}_Y] \circ [\mathrm{id}_X \otimes \beta_Y(Z)]$

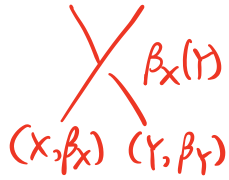
The tensor unit is $(1, \beta_1)$ where $\beta_1(X) := \mathrm{id}_X$.



The composition and tensor product of morphisms in $Z(\mathcal{C})$ are inherite from $\mathcal{C}$.

The braiding in $Z(\mathcal{C})$ is given by

$$b_{(X,\beta_X),(Y,\beta_Y)} := \beta_X(Y)$$



$\Rightarrow Z(\mathcal{C})$ is a braided monoidal cat.

Rem. 1. $\mathcal{C}$ is monoidal $\xrightarrow{Z(\cdot)}$ $Z(\mathcal{C})$ is braided monoidal

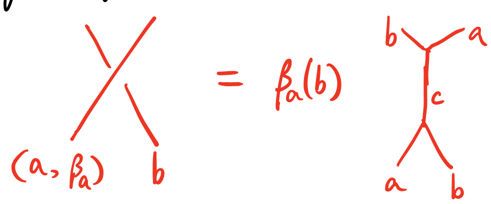
2. If $\mathcal{C}$ is fusion, then $Z(\mathcal{C})$ is modular, and $\dim Z(\mathcal{C}) = (\dim \mathcal{C})^2$, where $\dim \mathcal{C} := \sum_a d_a^2$

3. If $\mathcal{C}$ is modular tensor cat, then

$$Z(\mathcal{C}) = \mathcal{C} \boxtimes \mathcal{C}^{op.}$$

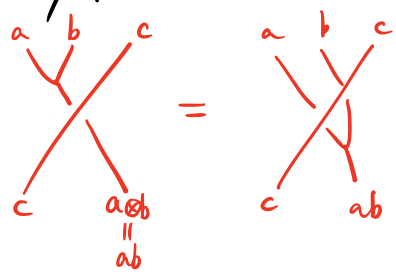
($\mathcal{C}^{op.}$ ↓ same as $\mathcal{C}$ with inverse braiding)

4. String diagram in simple object basis.

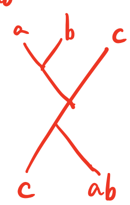


($N_c^{ab} = 0,1$)
$a \otimes b = c = ab$ is simple.

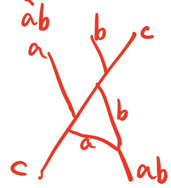
naturality:

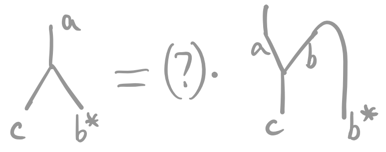


LHS $= \beta_c(a\otimes b)$

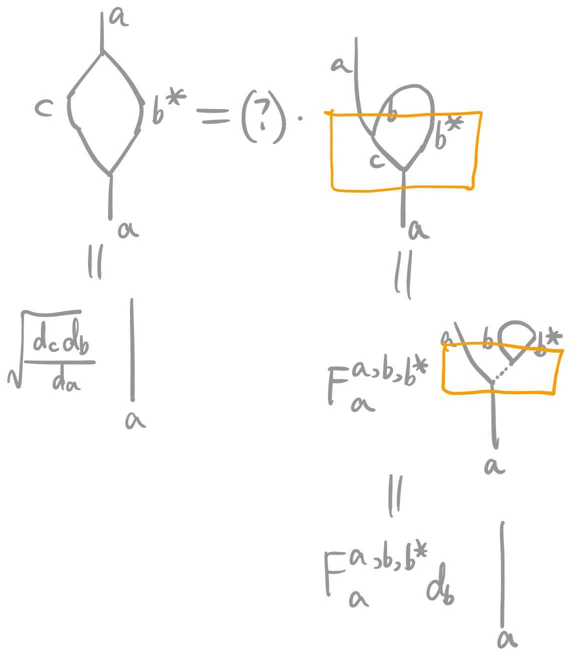


RHS $= \beta_c(a)\,\beta_c(b)$





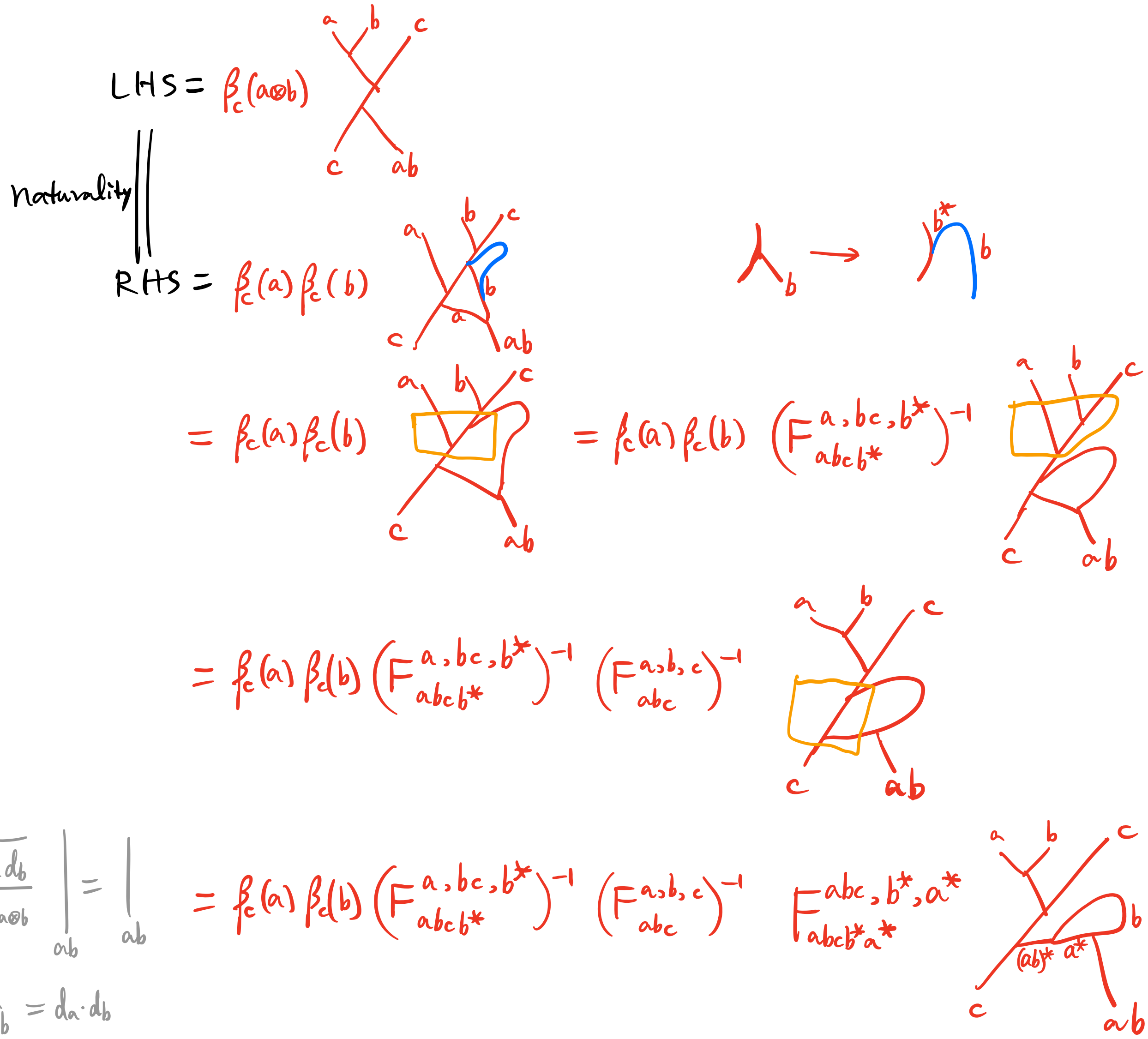
$$\mathrm{Hom}(c \otimes b^*, a) \cong \mathrm{Hom}(c, a \otimes b)$$

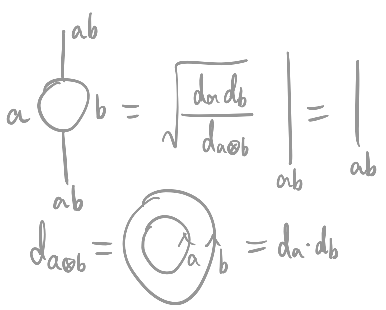


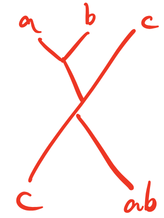
$$(?) = \sqrt{\frac{d_a}{d_b d_c}}\, d_b\, F_a^{abb^*} = \sqrt{\frac{d_a d_b}{d_c}}\, F_a^{abb^*}$$

assume

$$1$$





$= \beta_c(a)\,\beta_c(b)\,\left(F^{a,bc,b^*}_{abcb^*}\right)^{-1}\left(F^{a,b,c}_{abc}\right)^{-1}F^{abc,b^*,a^*}_{abcb^*a^*}$ 

$$\Rightarrow \quad \beta_c(a\otimes b) = \beta_c(a)\,\beta_c(b)\,\left(F^{a,bc,b^*}_{abcb^*}\right)^{-1}\left(F^{a,b,c}_{abc}\right)^{-1}F^{abc,b^*,a^*}_{abcb^*a^*}$$

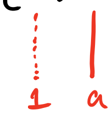
$\downarrow$

$\beta_c(-)$ half braiding of c.

5. Example of $Z(\mathrm{Vec}^{\omega_3}_{\mathbb{Z}_2})$: toric code. $\mathbb{Z}_2$ gauge theory.

$\omega_3 \to$ nontrivial $\in H^3(\mathbb{Z}_2, U(1))$ double semion model

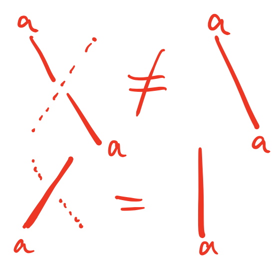
$\mathcal{C} = \mathrm{Vec}_{\mathbb{Z}_2}$ obj. $= \{1, a\}$, $a\otimes a = 1$



$\beta_1(1)$, $\beta_1(a)$; $\beta_a(1)$, $\beta_a(a)$

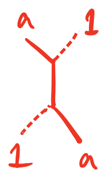
$\beta_1(1) = 1$, $\beta_a(1) = 1$

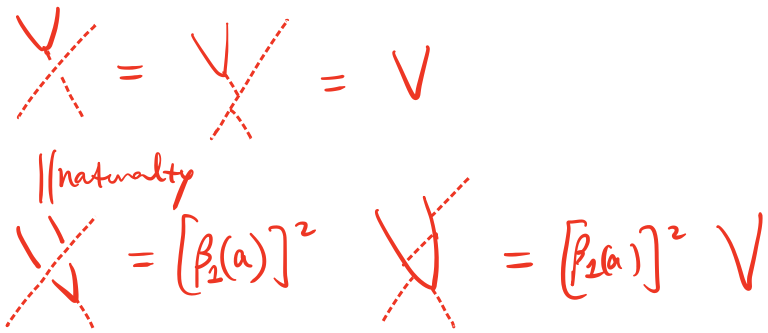
$\beta_1(a)$: ≠ ; $\beta_a(1)$: =



① $\beta_1(a)$:

$= \beta_1(a)\cdot$

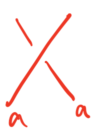




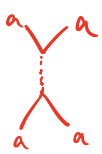
|| naturalty

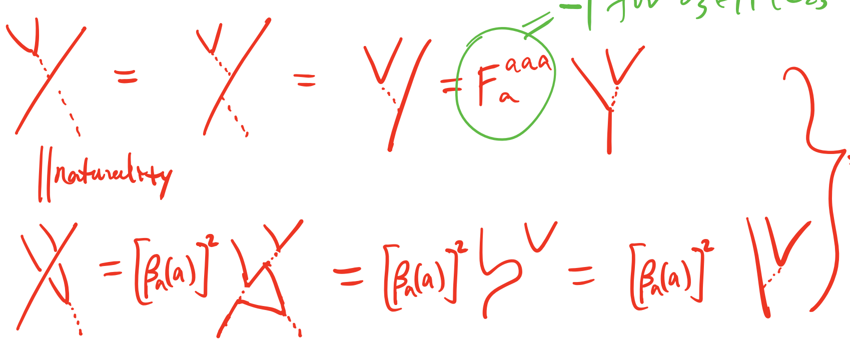
$= [\beta_1(a)]^2$ $= [\beta_1(a)]^2$ $\Rightarrow \beta_1(a) = \pm 1$

② $\beta_a(a)$:



$= \beta_a(a)$





$= F^{aaa}_a$ — -1 for $\omega_3 \in H^3(\mathbb{Z}_2, U(1))$

|| naturality

$= [\beta_a(a)]^2 = [\beta_a(a)]^2 = [\beta_a(a)]^2$ $\Rightarrow \beta_a(a) = \pm 1$

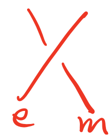
(semion) $\beta_a(a) = \pm i$ for $\omega_3 \in H^3$

$\Rightarrow$ There are 4 objects (X, β_X) in $Z(\mathrm{Vec}_{\mathbb{Z}_2})$

$$\underbrace{\left(1, \begin{smallmatrix}\beta_1(1)=1 \\ \beta_1(a)=+1\end{smallmatrix}\right)}_{1 \in Z(\mathrm{Vec}_{\mathbb{Z}_2})}, \quad \underbrace{\left(1, \begin{smallmatrix}\beta_1(1)=1 \\ \beta_1(a)=-1\end{smallmatrix}\right)}_{e}, \quad \underbrace{\left(a, \begin{smallmatrix}\beta_a(1)=1 \\ \beta_a(a)=+1\end{smallmatrix}\right)}_{m}, \quad \underbrace{\left(a, \begin{smallmatrix}\beta_a(1)=1 \\ \beta_a(a)=-1\end{smallmatrix}\right)}_{f}$$

$\Rightarrow$ Toric code $=$ $\mathbb{Z}_2$ gauge theory has 4 excitations.

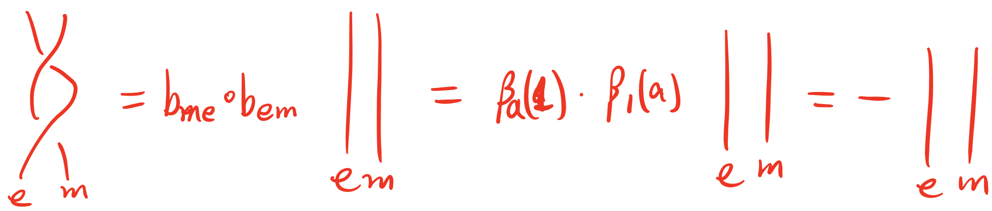
Statistics :



$$b_{e,m} = b_{\underbrace{(1,\beta_1(a)=-1)}_{e}, \underbrace{(a,\beta_a(a)=+1)}_{m}} = \beta_1(a) = -1$$

$$b_{(X,\beta_X),(Y,\beta_Y)} := \beta_X(Y)$$

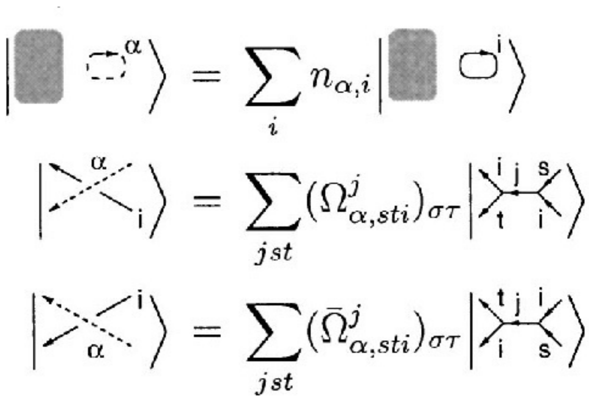
$$b_{m,e} = b_{(a,\beta_a(a)=+1),(1,\beta_1(a)=-1)} = \beta_a(1) = 1$$

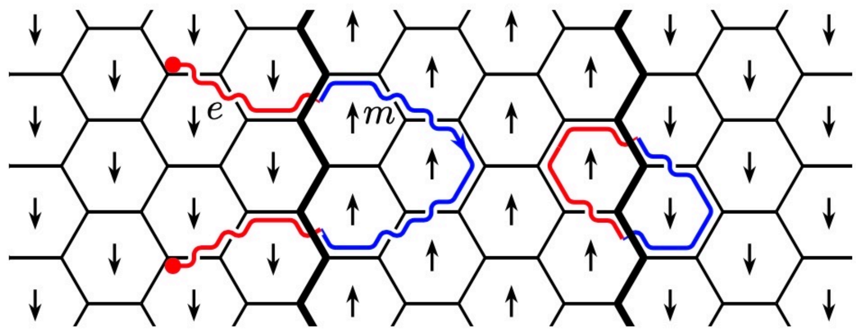


4.3. Excitation in Levin-Wen model of fusion cat. $\mathcal{C}$.

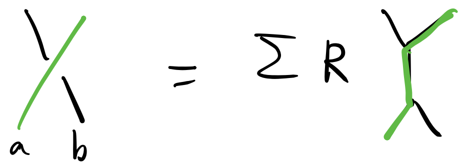
↳ described by $Z(\mathcal{C})$

(1) diagram for ribbon operators.

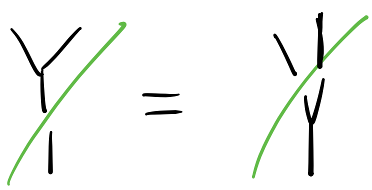




ribbon op. are labelled by obj. in $\mathcal{C}$.



[ribbon op., A_s, B_p] = 0 if $s, p \notin \partial$ ribbon.



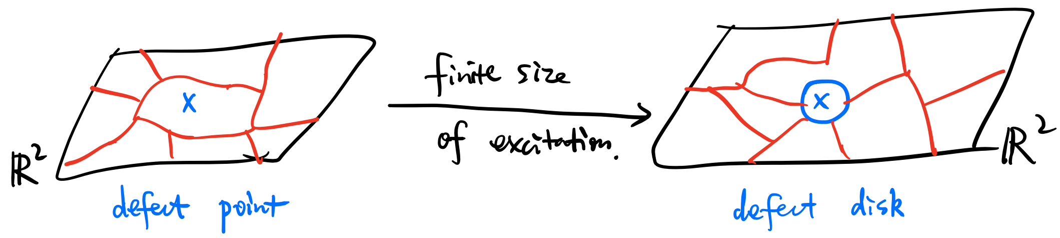
naturality

...

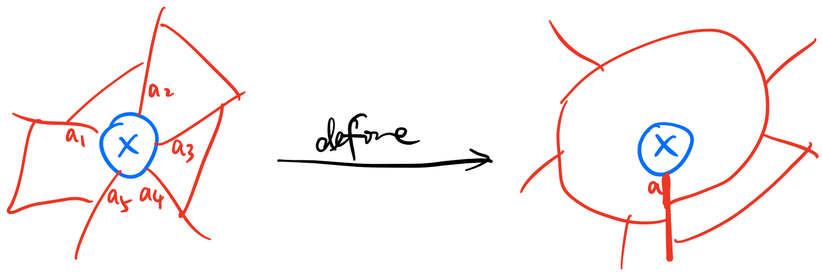
⟹ ribbon op. are labelled by $Z(\mathcal{C})$.

(2) Tube algebra approach

excitation = defect of the wave function or Hamiltonian.



String diagram on $\mathbb{R}^2 \setminus D^2$

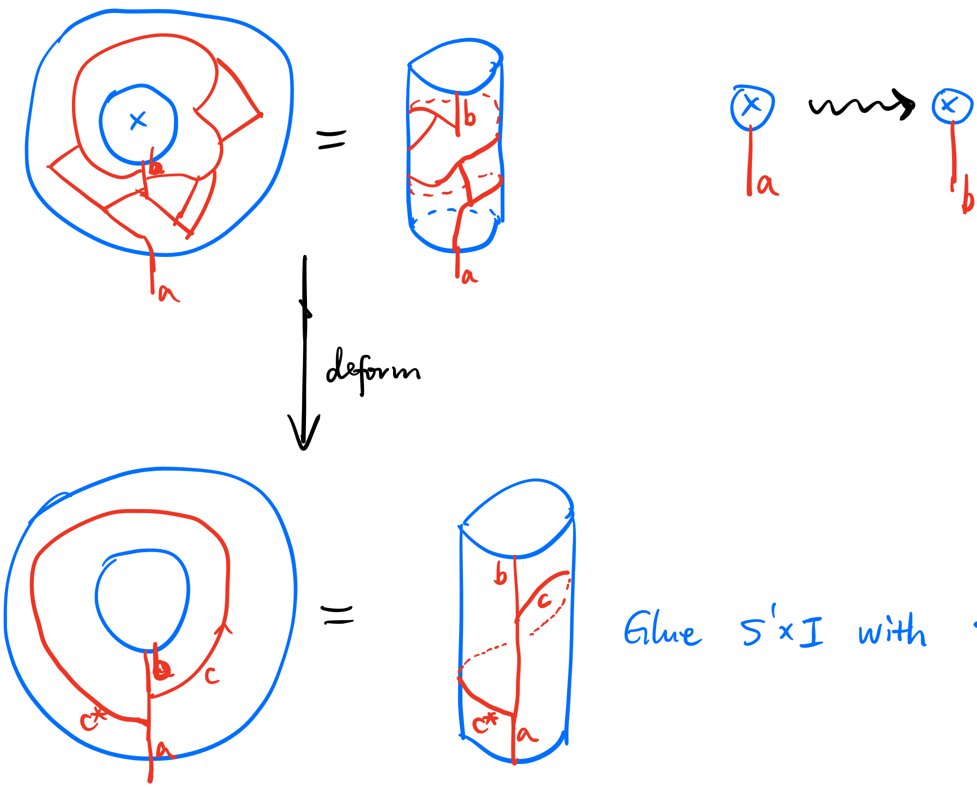


$\Rightarrow$ excitations are labelled by obj. $a \in \mathcal{C}$.

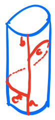
Q: What is the morphism between excitations?

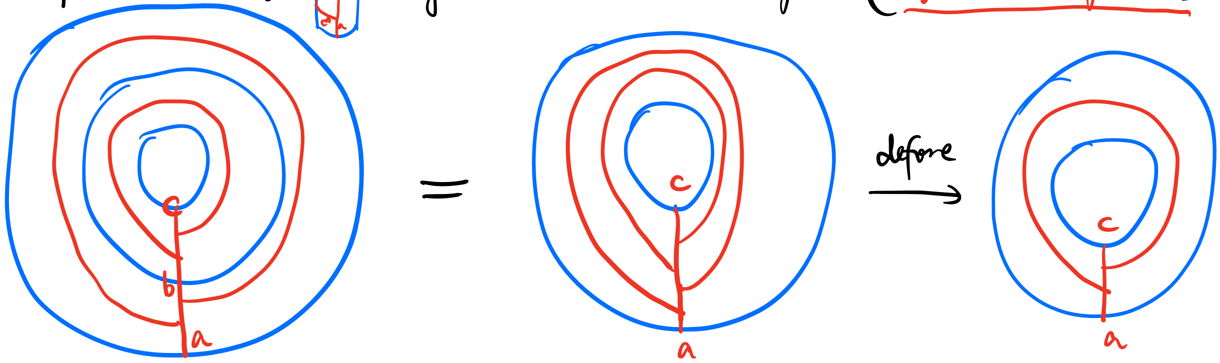
$\longleftrightarrow$ How to change one excitation into another?

A: Glue a cylinder $S^1 \times I$ with string diagram.



Glue $S^1 \times I$ with S^1 gives another S^1.

Composition of  gives us an algebra (Tube algebra)

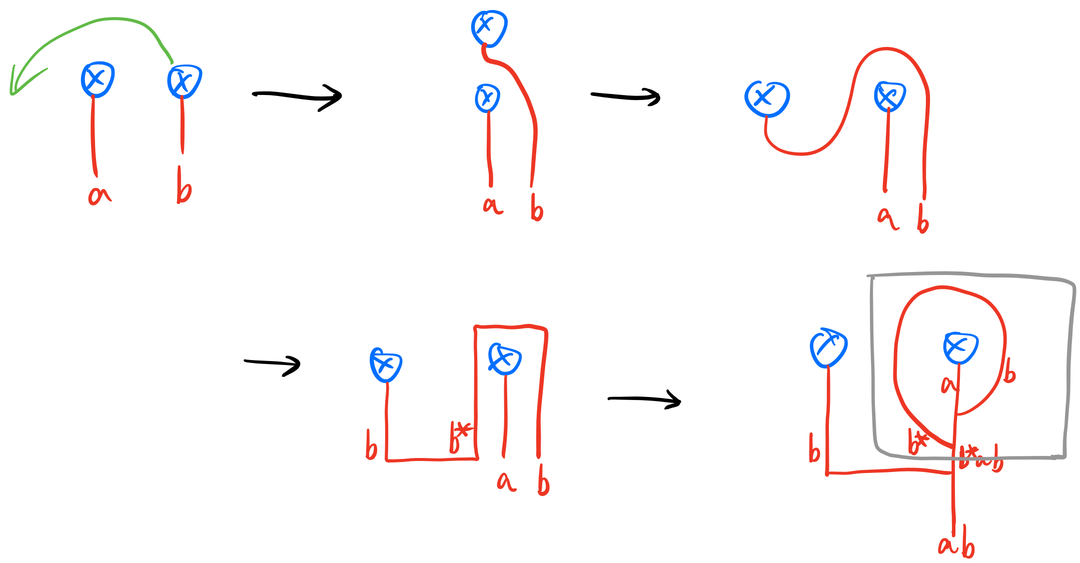


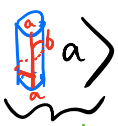
⇒ Excitations can be acted by tube algebra

⇒ Excitations are reps of tube algebra } ⇒ excitations $\in Z(\mathcal{C})$

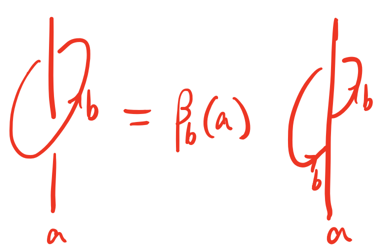
Cat. of rep. of tube algebra $= Z(\mathcal{C})$

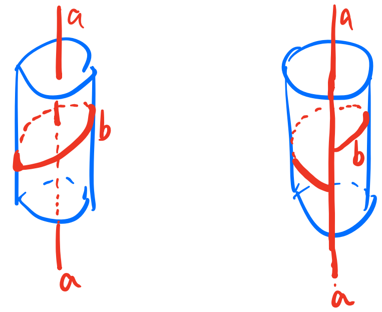
Braiding of excitations = defects :



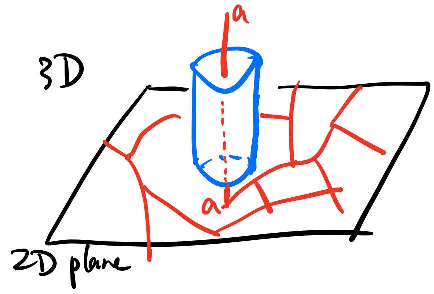
$|a, b\rangle \xrightarrow{\text{braiding}} |b,$  $a\rangle$

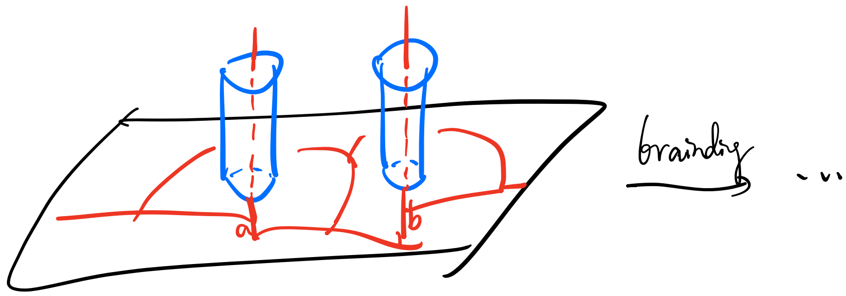
⇝ $|(a, \beta_a), (b, \beta_b)\rangle \xrightarrow{\text{braiding}} |(b, \beta_b), \beta_b(-)\,(a, \beta_a)\rangle$





Excitations can be understood as





$\Rightarrow$ Excitations should be labelled by $(a, \beta_a) \in Z(\mathcal{C})$.