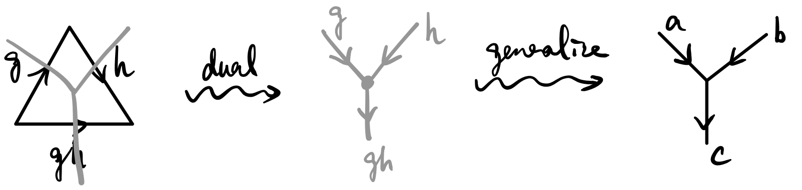
Fusion Categories and $\underbrace{\text{Turaev-Viro}}_{\text{spacetime}}$-$\underbrace{\text{Levin-Wen}}_{\text{space}}$ Model

Motivation. Generalize TQDM of a group G.

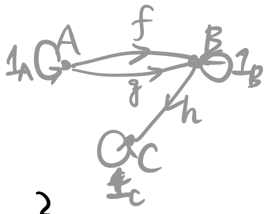
Hilbert space:



$g,h \in G.$ $\qquad\qquad$ $a,b,c \in \mathcal{C}$

$$\mathcal{C} \begin{cases} \text{types: } a,b,c,\dots \\ \text{multiplication / tensor product} \\ \qquad\qquad\qquad a\times b \end{cases}$$

3.1. Categories



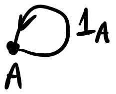
Def. A category $\mathcal{C}$ consists of

- A collection of objects $Obj(\mathcal{C}) = \{A, B, \cdots\}$
- A collection of morphisms $Hom(A,B)$ for $\forall A,B \in Obj(\mathcal{C})$
- A composition map for $\forall A,B,C \in Obj(\mathcal{C}) = \mathcal{I}$

$$C_{A,B,C}: \quad Hom(A,B) \times Hom(B,C) \longrightarrow Hom(A,C)$$
$$(f\quad,\quad g) \quad\longmapsto\quad g\circ f$$

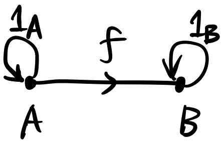
$$A \xrightarrow{f} B \xrightarrow{g} C \quad\longmapsto\quad A \xrightarrow{g\circ f} C$$

- An identity morphism $1_A = id_A \in Hom(A,A)$



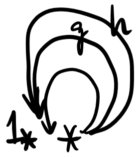
such that

- $(f\circ g)\circ h = f\circ(g\circ h)$ $\qquad$ associativity
- $f\circ 1_A = f = 1_B\circ f$ $\qquad$ identity



Examples (1) Set : $\text{set } A \xrightarrow{\text{map } f} \text{set } B$

(2) a group G :



$\text{Obj}(\mathscr{C}) = \{*\}$

$\text{Hom}(*,*) = G$

$$* \xrightarrow[g]{} * \xrightarrow[h]{} * \quad = \quad * \xrightarrow[g\cdot h]{} *$$

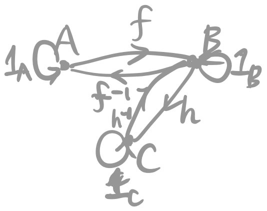
$$1_* = e \in G$$

Def: An isomorphism $f : A \to B$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

$$\begin{cases} g \circ f = 1_A \\ f \circ g = 1_B \end{cases}$$

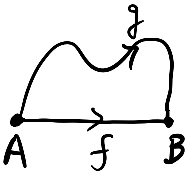
(3) groupoid : A cat $\mathscr{C}$ is a groupoid if every morphism is isomorphism.

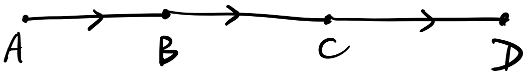


(4) fundamental groupoid of a topological space M :

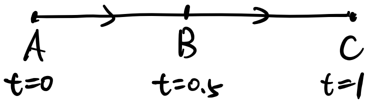
$\text{obj} = \{\text{points in } M\}$

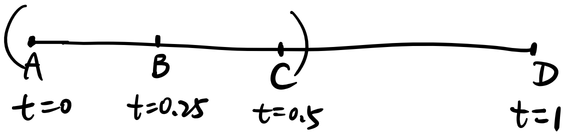
$\text{Hom}(A,B) = \{\text{Paths from } A \text{ to } B\} / \text{homotopy equivalence}$

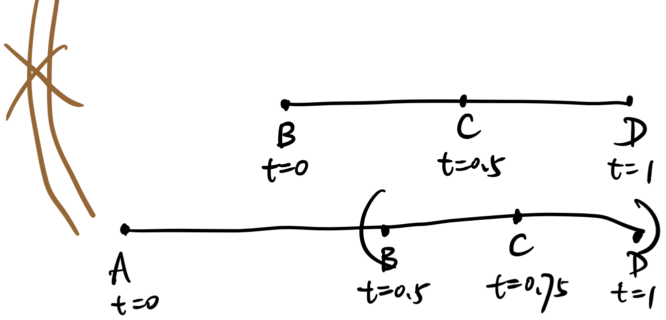




path: $I = [0,1] \to M$







(5) Grp : group G $\xrightarrow[\text{homomorphisms}]{\text{group}}$ group G'

(6) Vec : vector space V $\xrightarrow[\text{maps}]{\text{linear}}$ vector space W

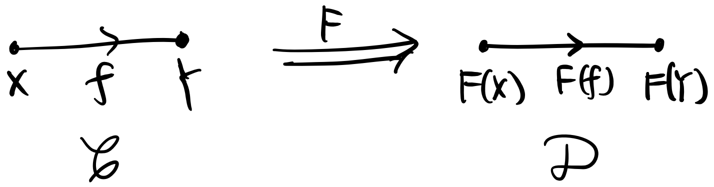
(7) Top : topological space M $\xrightarrow[\text{maps}]{\text{continuous}}$ topological space N

Def A functor from categories $\mathcal{C}$ to $\mathcal{D}$ is a map sending

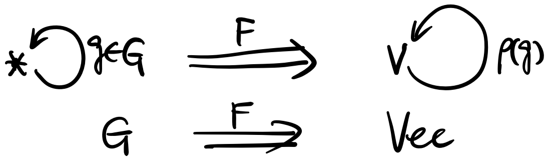
- any object $X \in \mathcal{C}$ to an object Y in $\mathcal{D}$.
- any morphism $f: X \to Y$ in $\mathcal{C}$ to a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$.

such that

- F preserves identity : $F(1_X) = 1_{F(X)}$
- F preserves composition : $F(gh) = F(g)\,F(h)$



Examples. (1)



representation of G.

(2) $H_n : \mathbf{Top} \longrightarrow \mathbf{Abel}$

$$\begin{tikzcd} M \arrow[d, "f"'] & & H_n(M) \arrow[d, "f_*"] \\ N \arrow[rr, phantom, "\overset{F}{\Longrightarrow}"] & & H_n(N) \end{tikzcd}$$

Def. Given two functors $F, G : \mathscr{C} \to \mathscr{D}$, a natural transformation $\alpha : F \Rightarrow G$ assigns to every object X in $\mathscr{C}$ a morphism $\alpha_X : F(X) \to G(X)$ in $\mathscr{D}$, s.t.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

$$\begin{tikzcd} & & F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] \arrow[rd, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\alpha_Y"] \\ X \arrow[r, "f"] & Y \arrow[ru, Rightarrow, "F"] \arrow[r, Rightarrow, "G"'] & G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

Example. a group G, given two rep (functors) $\rho : G \to V$
$\rho' : G \to V'$

a natural trasformation (intertwiner) is a map $f : V \to V'$.

s.t. $f \circ \rho(g) = \rho'(g) \circ f$ for $g \in G$.

$$\begin{tikzcd} & & V \arrow[r, "\rho(g)"] \arrow[d, "f = \alpha_*"'] \arrow[rd, phantom, "\circlearrowright"] & V \arrow[d, "f"] \\ * \arrow[r, "g"] & * \arrow[ru, Rightarrow, "\rho"] \arrow[r, Rightarrow, "\rho'"'] & V' \arrow[r, "\rho'(g)"'] & V' \end{tikzcd}$$

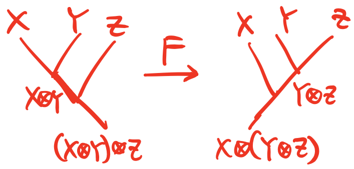
3.2. Fusion categories

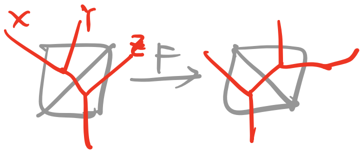
Def. A monoidal category consists of

- a category $\mathcal{C}$
- a tensor product functor $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$
- a unit object $1\in\mathcal{C}$
- a natural isomorphism

$$a_{X,Y,Z} : (X\otimes Y)\otimes Z \to X\otimes(Y\otimes Z),\ \forall X,Y,Z\in\mathcal{C}$$

↑
associator
F move





- natural isomorphism (left/right unitors) for $X\in\mathcal{C}$

$$l_X : 1\otimes X\to X$$

$$r_X : X\otimes 1\to X$$

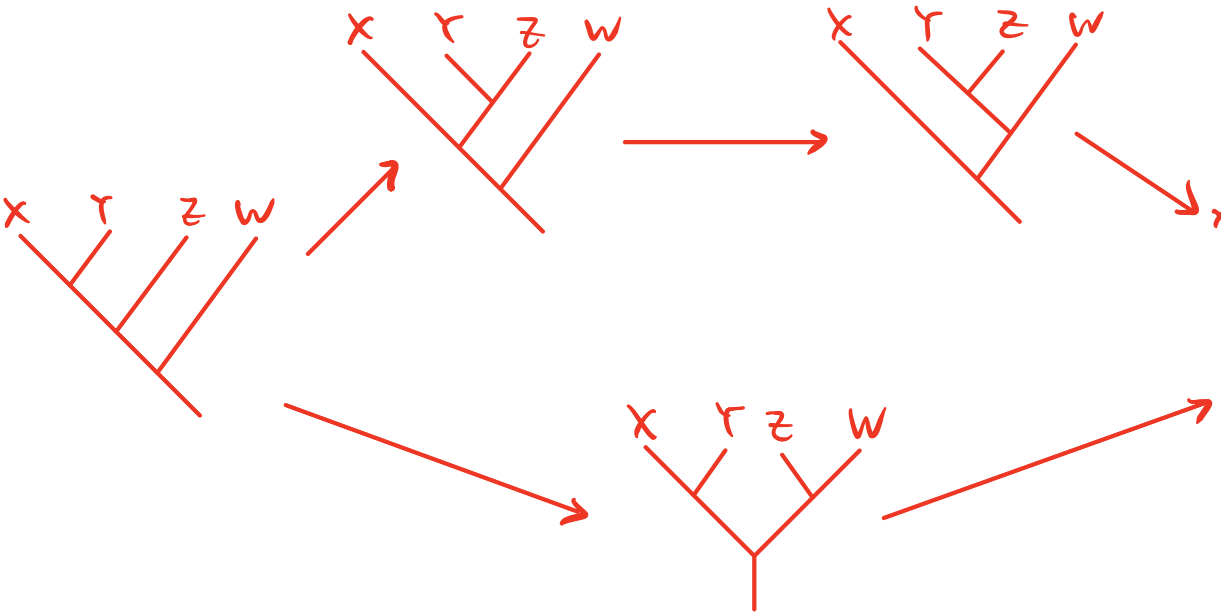
such that

-

$$\begin{tikzcd}
(X\otimes 1)\otimes Y \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "r_X\otimes 1_Y"'] & \arrow[d, phantom, "\circlearrowleft"] & X\otimes(1\otimes Y) \arrow[dl, "1_X\otimes l_Y"] \\
& X\otimes Y &
\end{tikzcd}$$

- pentagon equation:

$$\begin{tikzcd}
& (X\otimes(Y\otimes Z))\otimes W \arrow[rr, "a_{X,Y\otimes Z,W}"] & & X\otimes((Y\otimes Z)\otimes W) \arrow[dr, "1_X\otimes a_{Y,Z,W}"] & \\
((X\otimes Y)\otimes Z)\otimes W \arrow[ur, "a_{X,Y,Z}\otimes 1_W"] \arrow[drr, "a_{X\otimes Y,Z,W}"'] & & \arrow[phantom, "\circlearrowleft"] & & X\otimes(Y\otimes(Z\otimes W)) \\
& & (X\otimes Y)\otimes(Z\otimes W) \arrow[urr, "a_{X,Y,Z\otimes W}"'] & &
\end{tikzcd}$$


X Y Z W
X Y Z W
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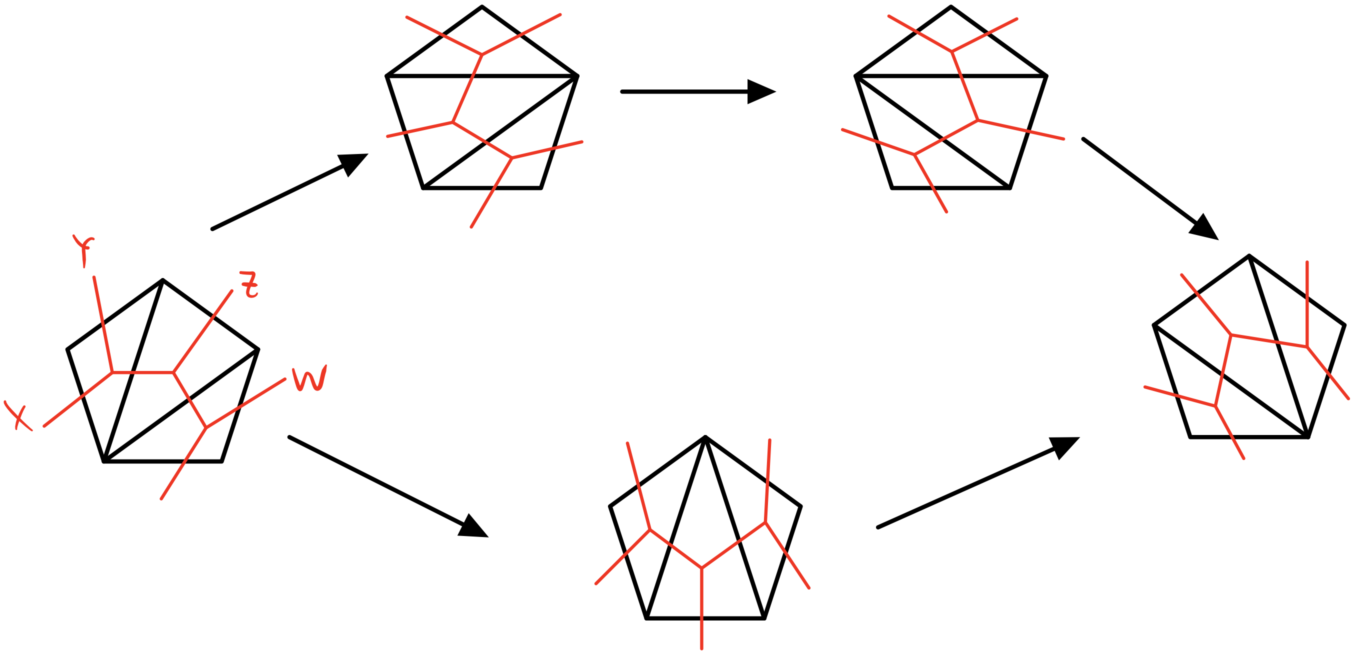

Y
Z
X
W

Examples (1) $\mathrm{Rep}(G)$. obj = vector space

Hom = G-invariant linear maps.

$V \otimes W$

1 = trivial rep on $\mathbb{C}$.

$a_{x,y,z}$ is trivial

(2) Vec_G. obj = V_g for $\forall g \in G$

$$\mathrm{Hom}(V_g, V_h) = \begin{cases} \mathbb{C}, & g = h \\ 0, & g \neq h \end{cases}$$

$$V_g \otimes V_h = V_{gh}$$

a_{V_g, V_h, V_k} is trivial

$1 = V_e$, e identity element in G.

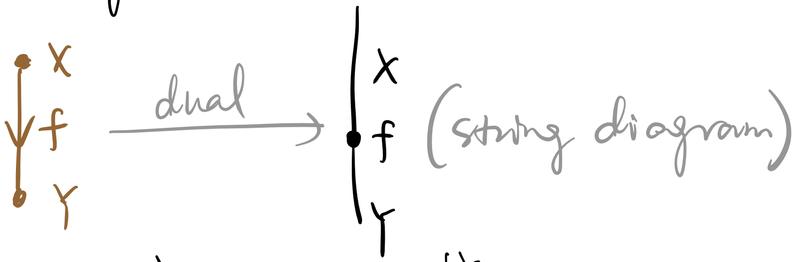
(3) $\mathrm{Vec}_G^{\nu_3}$: $a_{V_g, V_h, V_k} := \nu_3(g,h,k) \in U(1)$

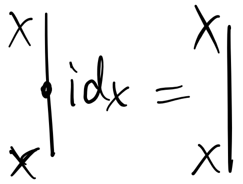
pentagon eq $\Leftrightarrow d\nu_3 = 1$

$\Leftrightarrow \nu_3 \in Z^3(G, U(1))$

Def. A **fusion category** is a $\overset{\overset{\text{with dual}}{\uparrow}}{\text{rigid}}$, $\overset{\overset{a \oplus b}{\uparrow}}{\text{semisimple}}$, $\overset{\overset{\text{hom set is vector space.}}{\uparrow}}{\text{linear}}$ monoidal category with finitly many isomorphism classes of simple objects such that $\mathrm{Hom}(1, 1) = \mathbb{C}$.

3.3. String diagram





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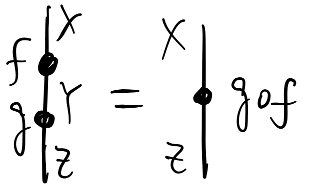
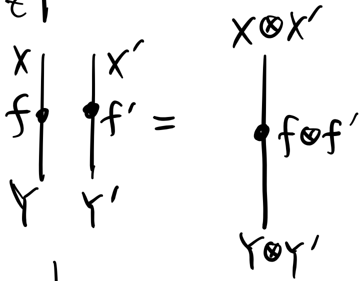
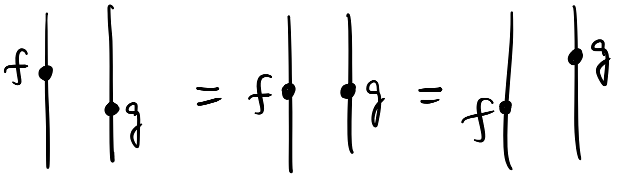
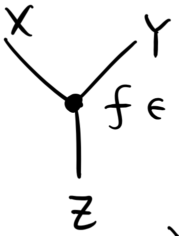


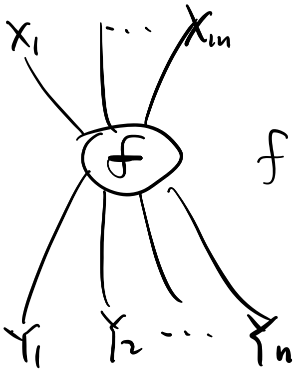
tensor: $X \otimes Y$





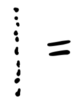


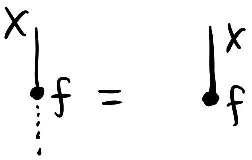
$f \in Hom(X \otimes Y, Z)$



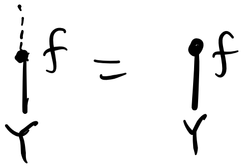
$f \in Hom(X_1 \otimes \cdots \otimes X_m, Y_1 \otimes \cdots \otimes Y_n)$

identity 1 :



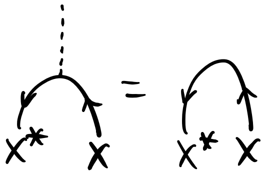


$f \in Hom(X, 1)$

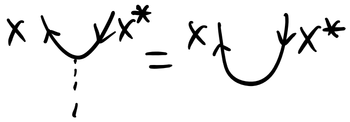


$f \in Hom(1, Y)$

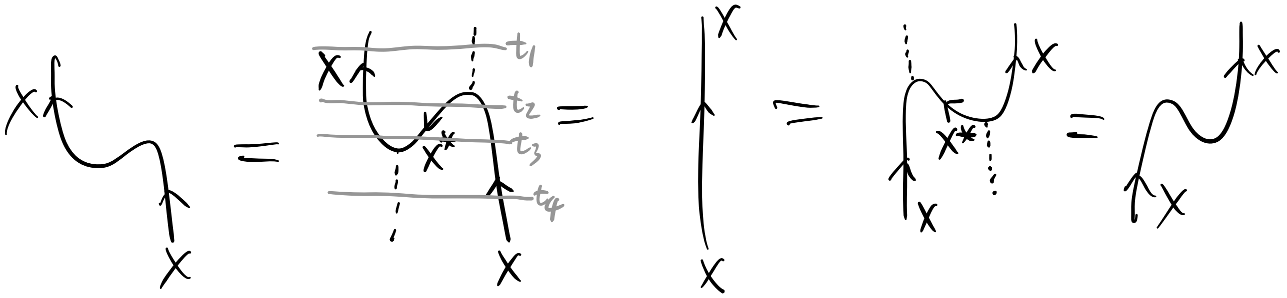
duals : unit $i_X : 1 \longrightarrow X^* \otimes X$



counit $e_X : X \otimes X^* \longrightarrow 1$

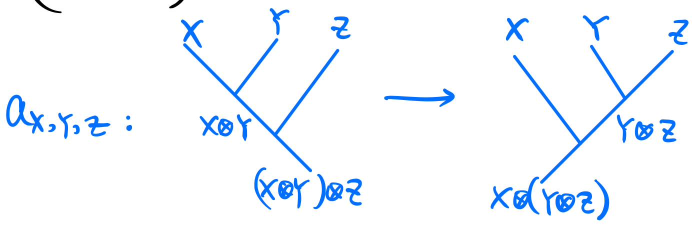


s.t.



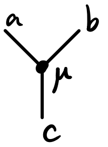
$$\begin{tikzcd}
\underset{t_1}{X\otimes 1} \arrow[r, "id_X\otimes i_X"] \arrow[drr, "r_X"'] & \underset{t_2}{X\otimes(X^*\otimes X)} \arrow[r, "a^{-1}_{X,X^*,X}"] \arrow[dr, phantom, "\circlearrowright"] & \underset{t_3}{(X\otimes X^*)\otimes X} \arrow[r, "e_X\otimes id_X"] & \underset{t_4}{1\otimes X} \arrow[dl, "l_X"] \\
& & X &
\end{tikzcd}$$

associator (F move):



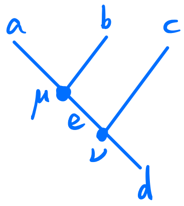
In simple object basis :

$V^{ab}_c := Hom(a\otimes b, c) \ni$

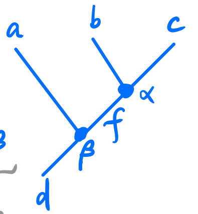


$\longleftrightarrow |a,b;c,\mu\rangle$

$N^{ab}_c := \dim V^{ab}_c$



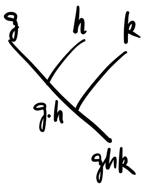
$$= \sum_{f,\alpha,\beta} \Big(F^{\overset{\text{parameters}}{abc}}_{d}\Big)_{\underbrace{e\mu\nu}_{\text{row}},\underbrace{f\alpha\beta}_{\text{col}}}$$



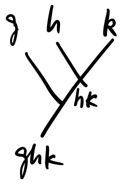
$$\sum_e N^{ab}_e N^{ec}_d = \sum_f N^{bc}_f N^{af}_d$$

unitary fusion category: F is unitary.

$\mathcal{C} = \mathrm{Vec}_G^{\nu_3}$:

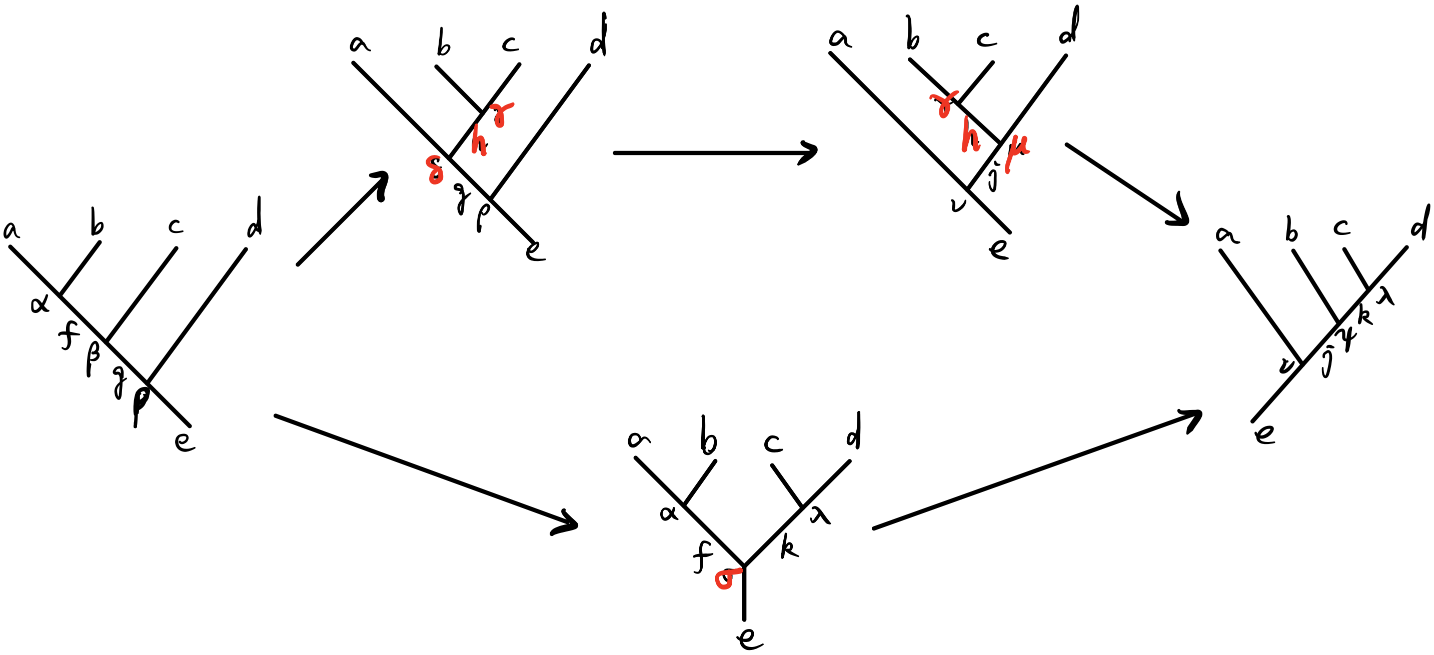


$= \nu_3(g,h,k)$



$$\nu_3(g,h,k) := \left(F^{g,h,k}_{ghk}\right)_{1,1}$$

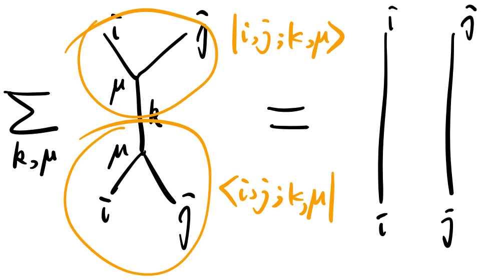
pentagon eq:



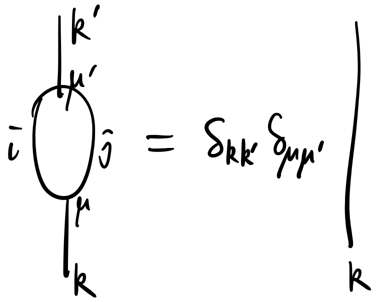
$$\sum_{h;\gamma,\delta,\mu} \left(F^{abc}_{g}\right)_{f\alpha\rho,h\gamma\delta} \left(F^{ahd}_{e}\right)_{g\delta\rho,j\mu\nu} \left(F^{bcd}_{j}\right)_{h\gamma\mu,k\lambda\psi}$$

$$= \sum_{\sigma} \left(F^{fcd}_{e}\right)_{g\beta\rho,k\lambda\sigma} \left(F^{abk}_{e}\right)_{f\alpha\sigma,j\psi\nu}$$

important relation:



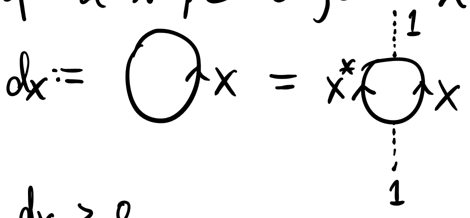
$$\sum_{k,\mu} |ij;k\mu\rangle\langle ij;k\mu| = I$$



$\langle ij;k'\mu' | ij;k\mu\rangle = \delta_{kk'}\delta_{\mu\mu'}$

quantum dimension :

d_X of a simple object $X \in \mathcal{C}$ is

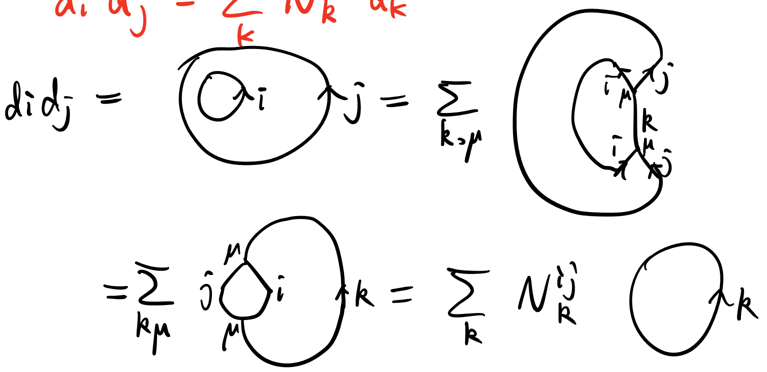


$\in \mathrm{Hom}(1,1) = \mathbb{C}$

$d_X > 0$.

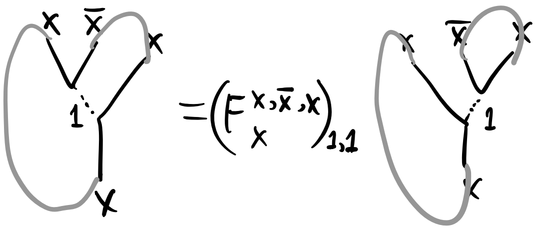
property : $d_i d_j = \sum_k N_k^{ij} d_k$

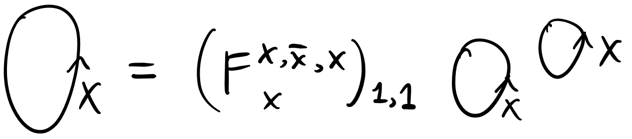
proof :



$$= \sum_k N_k^{ij} d_k$$

$d_X = \left[(F_X^{X,\bar{X},X})_{1,1} \right]^{-1}$





$$d_X = (F_X^{X,\bar{X},X})_{1,1} \, (d_X)^2$$

physical meaning of d_x :

$$N_k^{ij}\, d_k = d_i\, d_j$$

$$(N^i)_k^j\, d_k = d_i\, d_j$$

$$j\left(\;\cancel{N^i}\;\right)\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = d_i \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} j$$

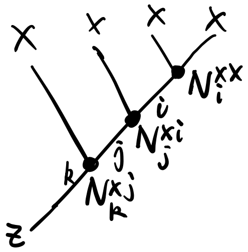
$\Rightarrow (d_1 \cdots d_n)^T$ is the eigenvector of N^i with eigenvalue d_i.

$N_k^{ij} \geq 0$, $d_i > 0$

$\Rightarrow$ d_i is the largest eigenvalue of N^i.

Q: what is dim of $(X)^{\otimes n}$, $n \to \infty$?

A:



$$\dim X^{\otimes n} := \dim\left[\bigoplus_z \mathrm{Hom}(X^{\otimes n}, z)\right]$$

$$\dim(X)^{\otimes n} = \sum_{i,j,k,\cdots} N_i^{XX}\, N_j^{Xi}\, N_k^{Xj} \cdots$$

$$= \sum_{i,j,k,\cdots} (N^X)_i^X\, (N^X)_j^i\, (N^X)_k^j \cdots$$

$$= \sum_z \left[(N^X)^n\right]_{x,z} \overset{n\to\infty}{\sim} (d_X)^n$$

$\Rightarrow$ $\dim(X^{\otimes n}) \sim d_X{}^n$ for $n \to \infty$

quantum dimension of X.

eg: $\dim\left(\overset{\bullet}{\text{Majorana}} \quad \overset{\bullet}{\text{Majorana}}\right) = 2$

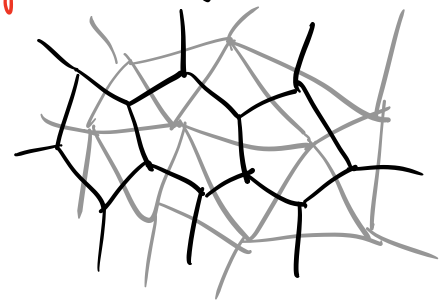
$\Rightarrow \dim\left(\overset{\bullet}{\text{Majorana}}\right) = \sqrt{2}$

Finally, $\forall$ trivalent graph

label $\begin{cases} \text{edge by } X \in Obj(\mathcal{C}) \\ \text{vertex by } Hom(X\otimes Y, Z) \text{ or } Hom(X, Y\otimes Z) \end{cases}$

$\updownarrow$ dual

triangulation of M_2

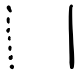


3.4. Examples.

Classification of fusion cat ?

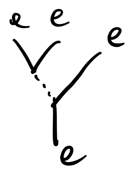
Very hard !!! $\mathcal{C} = Vec_G \to$ classification of finite simple groups.
(already very hard !)

(1) $obj(\mathcal{C}) = \{1, e\} = \mathbb{Z}_2$

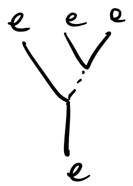


$e\otimes e = 1$, $e^* = e$

associator $a_{e,e,e}$:

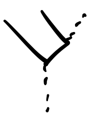


$= \lambda$

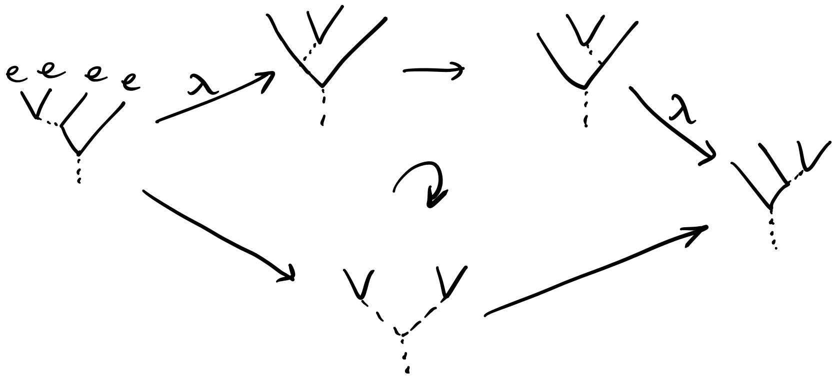


$a_{e,e,1}$:

$=$



pentagon:



$$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$$H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \ni \omega_3 \begin{cases} ① \ \omega_3(a,b,c) = 1 \ (\forall a,b,c) : & \lambda = 1 \quad \text{toric code} \\ ② \ \omega_3(e,e,e) = -1 : & \lambda = -1 \quad \text{double semion.} \end{cases}$$

(2) Fibonacci ·

- $\mathrm{Obj}(\mathcal{C}) = \{1, \tau\}$
- $\tau \otimes \tau = 1 \oplus \tau$ ⇝ non-Abelian

$N_1^{\tau\tau} = N_\tau^{\tau\tau} = 1$

$\tau^{\otimes n} = F_{n-2} 1 \oplus F_{n-1} \tau$ where $F_n = 1, 1, 2, 3, 5, 8, \cdots$ is the Fibonacci number.

$\Downarrow$

$$\tau \otimes (\tau^{\otimes n}) = \tau \otimes (F_{n-2} 1 \oplus F_{n-1} \tau) = F_{n-2} \tau \oplus F_{n-1} (\tau \otimes \tau)$$

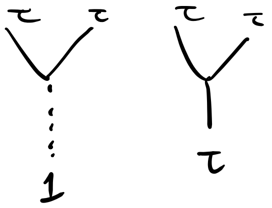
$$= F_{n-2} \tau \oplus F_{n-1} (1 \oplus \tau) = F_{n-1} 1 \oplus \underbrace{(F_{n-2} + F_{n-1})}_{F_n} \tau$$

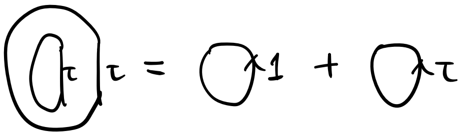
$$= F_{n-1} 1 \oplus F_n \tau$$

No class on 2021.10.20 and 2021.10.25 !

↙ Prof. Liang Kong's talk

- $\tau \otimes \tau = 1 \oplus \tau$

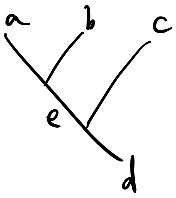


⇒


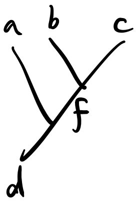
$$\Rightarrow \quad (d_\tau)^2 = 1 + d_\tau$$

$$\Rightarrow \quad d_\tau = \frac{1+\sqrt{5}}{2} = \phi \quad \text{Golden ratio}$$

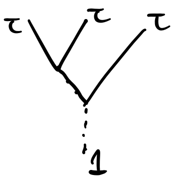
- F symbol.



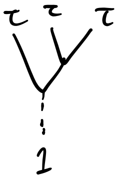
$$= \sum_f \left(F^{abc}_d\right)_{e,f}$$



① $\tau \otimes \tau \otimes \tau \longrightarrow 1$

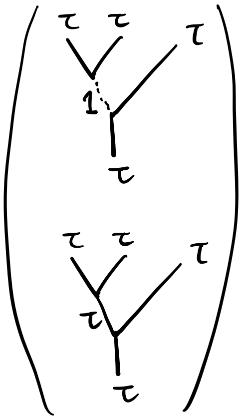


$$= \quad F^{\tau\tau\tau}_1$$

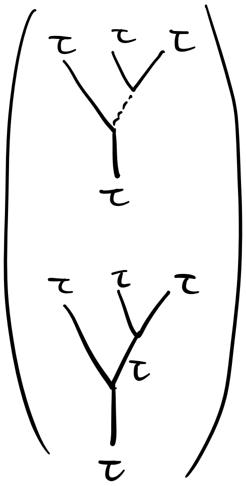


$F^{\tau\tau\tau}_1 = 1$

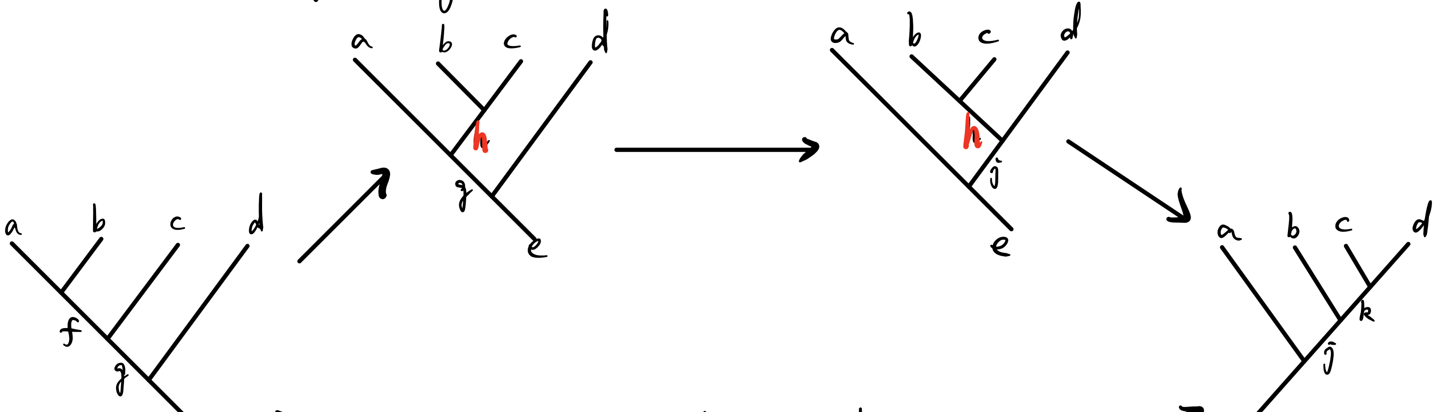
② $\tau \otimes \tau \otimes \tau \longrightarrow \tau$

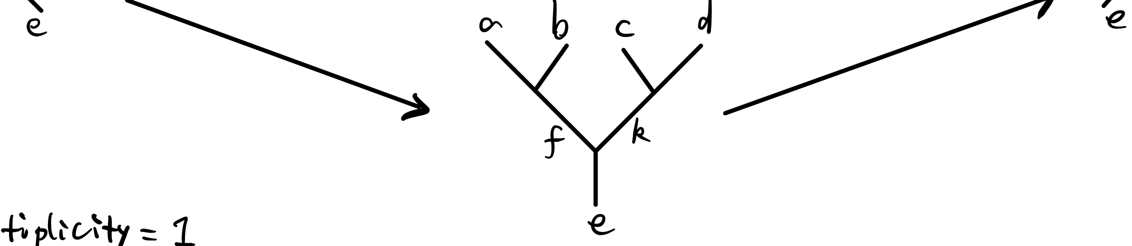


$$= \underbrace{\begin{pmatrix} \left(F^{\tau\tau\tau}_\tau\right)_{1,1} & \left(F^{\tau\tau\tau}_\tau\right)_{1,\tau} \\ \left(F^{\tau\tau\tau}_\tau\right)_{\tau,1} & \left(F^{\tau\tau\tau}_\tau\right)_{\tau,\tau} \end{pmatrix}}_{\begin{pmatrix} \phi^{-1} & \sqrt{\phi^{-1}} \\ \sqrt{-\phi^{-1}} & -\phi^{-1} \end{pmatrix}}$$



- pentagon eq:



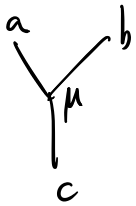


multiplicity = 1

$$\Downarrow$$

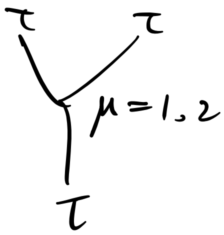
$$\sum_{h} \left(F_g^{abc}\right)_{f,h} \left(F_e^{ahd}\right)_{g,j} \left(F_j^{bcd}\right)_{h,k}$$

$$= \left(F_e^{fcd}\right)_{g,k} \left(F_e^{abk}\right)_{f,j}$$



$$\mu \in \mathrm{Hom}(a \otimes b, c) = \mathbb{C}^{(\text{multiplicity})}$$

If $\tau \otimes \tau = 1 \oplus \underset{\overset{\uparrow}{\mu=1}}{\tau} \oplus \underset{\overset{\nearrow}{\mu=2}}{\tau}$, then



3.5. Levin-Wen model

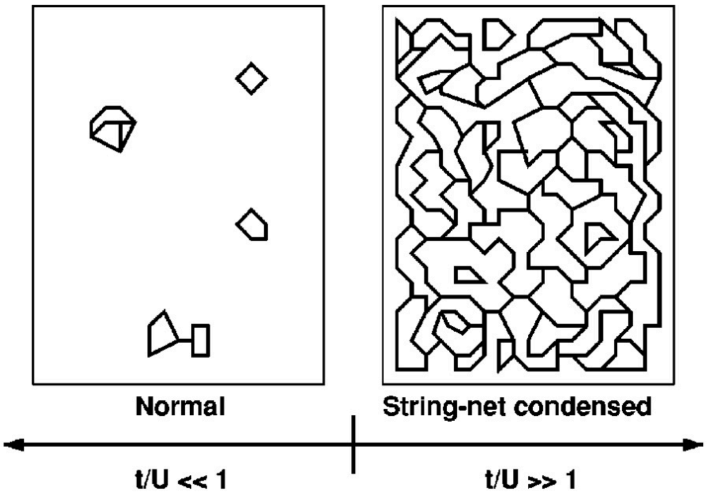
PRB 71, 045110 (2005)

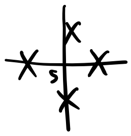
- string-net condensation

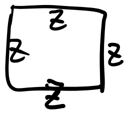
string-net → trivalent graph with fusion category labels

condensation → $\langle b\rangle \neq 0$ $\quad |\psi_{GS}\rangle = \sum_n \#\,(b^+)^n|0\rangle$

Σ (all possible string-net conf.)



Example : $H = -\sum_s$ 

$- t\sum_p$ 

$- U\sum_{\text{links}} X$

new string tension term

⋯⋯ $X=+1$

—— $X=-1$

① $t/U \ll 1$, $U\to+\infty$: $X=+1$ for all links

$|\psi_{GS}\rangle = \bigotimes_{\text{link } \ell} |X=+1\rangle_\ell$ → product state.

② $t/U \gg 1$, $U\to 0$: TC,

$|\psi\rangle = \Sigma$ (all closed loops) → string-net cond.

- string-net configuration :

color / label trivalent graph (V,E) by the fusion category $\mathcal{C}$.

$E \to \mathrm{obj}(\mathcal{C})$

$V \to \mathrm{Hom}_{\mathcal{C}}(i\otimes j, k)$

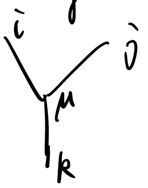
(1) string type:

simple object $i = 0, 1, \cdots, N$ (N finite)

(2) branching rule:

multiplicity $= 1$

$\Updownarrow$



$\mu \in \mathrm{Hom}(i \otimes j, k) \overset{\text{assume}}{=} \begin{cases} 0, & \text{if } i,j \text{ can not fuse into } k. \\ \mathbb{C}, & \text{.. can ... } k \end{cases}$

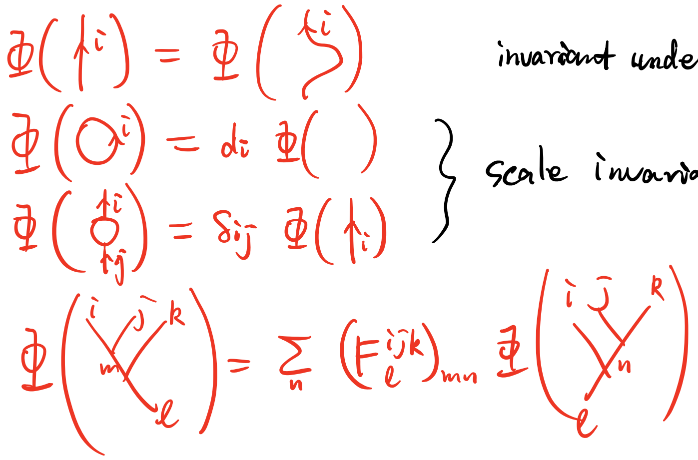
(3) string orientation:

$\uparrow i = \downarrow i^*$

- Ground state wave function

$$|\Psi_{GS}\rangle = \sum_{\substack{\text{string-net} \\ \text{conf. } X}} \Phi(X) \, |X\rangle$$

$\{\Phi(X)\}$ are related to each other under local move $X \to X'$:



invariant under ambient isotopy

scale invariance

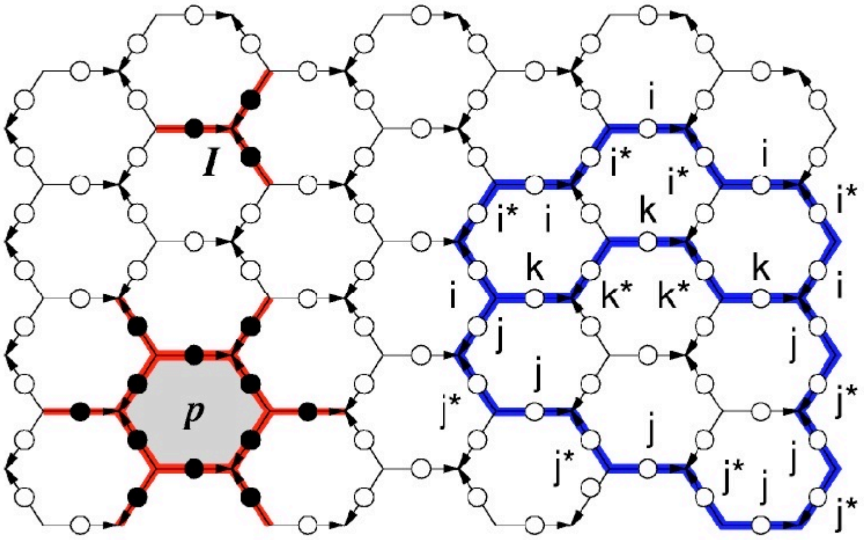
consistency condition: pentagon equation, rigid struture, ...

- Exactly solvable commuting-projector Hamiltonian

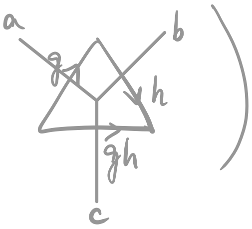
$$H = -\sum_s A_s - \sum_p B_p$$

constraint on conf.

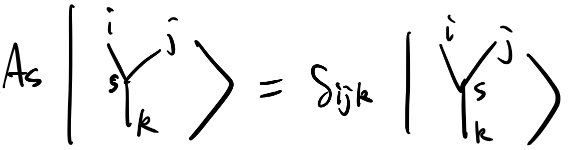
"gauge transf"



(dual to QDM :



)



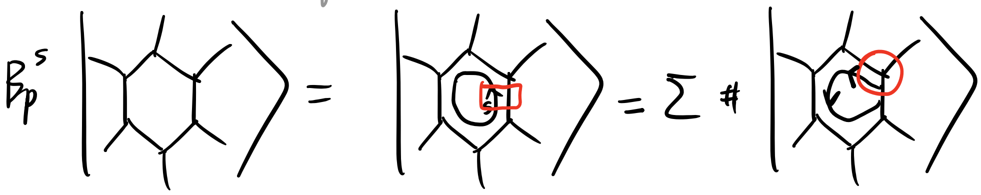
, $\delta_{ijk} = \begin{cases} 1, & N_k^{ij} \neq 0 \\ 0, & N_k^{ij} = 0 \end{cases}$

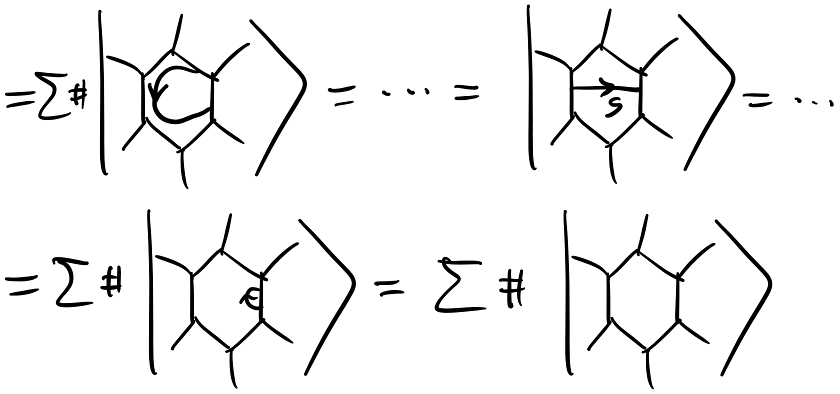
3-body interaction

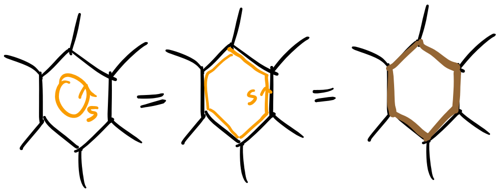
$$B_p = \frac{1}{D^2} \sum_{s \in obj(\mathcal{C})} d_s \cdot B_p^s \quad ,$$

q dim

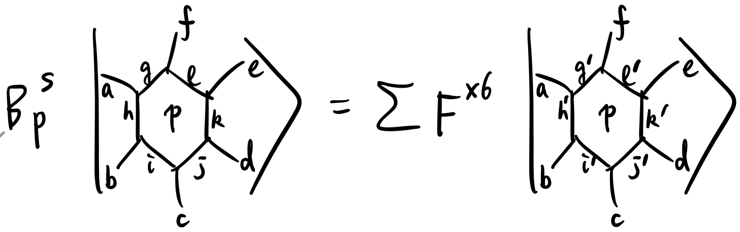
$D^2 := \sum_s d_s^2$ total q dim.







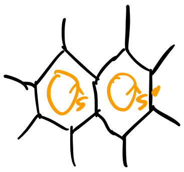
similar to $A_s = \frac{1}{|G|} \sum_g A_s^g$



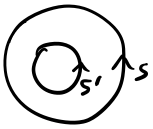
12-body interaction

properties :

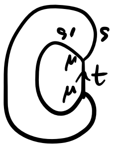
$$\left\{\begin{array}{l}[A_s, A_{s'}]=0\\ [B_p, B_{p'}]=0\\ [A_s, B_p]=0\\ A_s^2 = A_s\\ B_p^2 = B_p\end{array}\right.$$



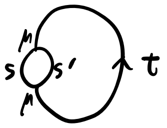
$$B_p^2 = \left(\frac{1}{D^2}\sum_s d_s \bigcirc_s\right)^2 = \frac{1}{D^4}\sum_{ss'} d_s d_{s'}$$

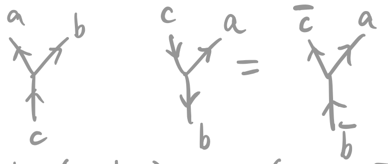


$$= \frac{1}{D^4}\sum_{ss'}\sum_{t\mu} d_s d_{s'}$$



$$= \frac{1}{D^4}\sum_{ss'}\sum_{t\mu} d_s d_{s'}$$



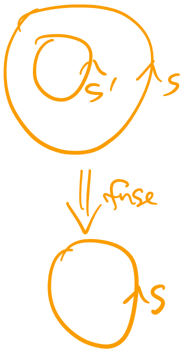


$\mathrm{Hom}(a\otimes b, c) \cong \mathrm{Hom}(\bar{c}\otimes a, \bar{b})$

$N^{ab}_c = N^{\bar{c}a}_{\bar{b}}$

$$= \frac{1}{D^4}\sum_{ss't} d_s d_{s'} N^{s's}_t \bigcirc_t$$

$B_p = \sum d_s \bigcirc_s$



$$= \frac{1}{D^4}\sum_{ss't} d_s d_{s'} N^{\bar{t}s'}_{\bar{s}} \bigcirc_t$$

$d_s = \bigcirc_s = {}_s\bigcirc = d_{\bar{s}}$

$$= \frac{1}{D^4}\sum_{s't} d_{s'}\left(\sum_{\bar{s}} N^{\bar{t}s'}_{\bar{s}} d_{\bar{s}}\right) \bigcirc_t$$

$d_i d_j = \sum_k N^{ij}_k d_k$

$$= \frac{1}{D^4}\sum_{s't} d_{s'}\left(d_{\bar{t}} d_{s'}\right) \bigcirc_t$$

$$= \frac{1}{D^2}\underbrace{\left[\frac{1}{D^2}\sum_{s'}(d_{s'})^2\right]}_{1} \sum_t d_t \bigcirc_t$$

$$= B_p$$

$\Rightarrow$ commuting-projector Hamiltonian

$$\Rightarrow \left\{\begin{array}{l} A_s|\Psi_{GS}\rangle = |\Psi_{GS}\rangle \rightarrow \text{valid string-net conf.}\\ B_p|\Psi_{GS}\rangle = |\Psi_{GS}\rangle \rightarrow \text{condensation/superposition of all possible conf.}\end{array}\right.$$

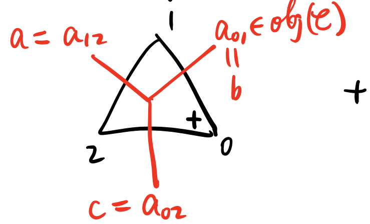
$$|\Psi_{GS}\rangle = \sum_{\substack{\text{string diag.}\\ X}} \Phi(X)\,|X\rangle$$

3.6. Turaev-Viro model = Partition function of Levin-Wen model

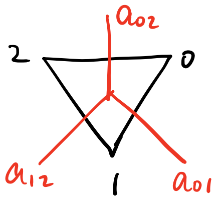
(1) Turaev-Viro - Barrett-Westbury invariants for 3-manifold

1992 for quantum $sl_2(\mathbb{C})$ 1996 for ∀ fusion cat.

- triangle

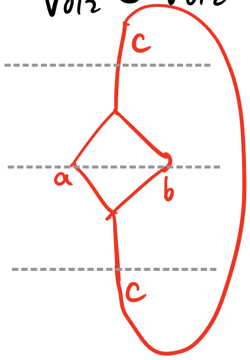


+ triangle ⟨012⟩ → vector space $V_{012} := \mathrm{Hom}_{\mathcal{C}}(a_{12} \otimes a_{01}, a_{02})$

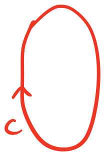


− triangle ⟨012⟩ → vector space $V^*_{012} := \mathrm{Hom}_{\mathcal{C}}(a_{02}, a_{12} \otimes a_{01})$

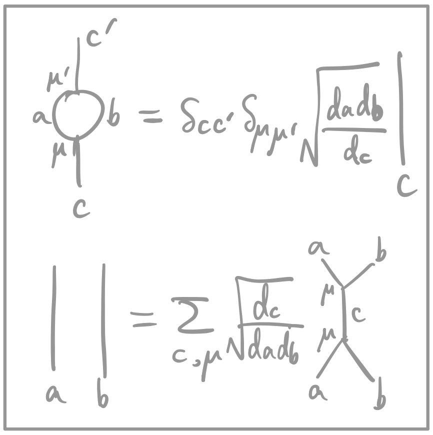
Pairing : $V_{012} \otimes V^*_{012} \longrightarrow \mathbb{C}$



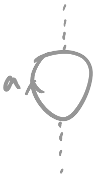
$= \sqrt{\frac{d_a d_b}{d_c}}$



$= \sqrt{d_a d_b d_c}$



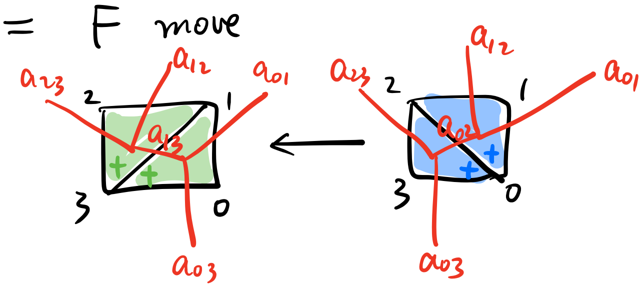
Special case $\begin{cases} a = b^* \\ c = 1 \end{cases}$:



$= d_a$

- tetrahedron = F move

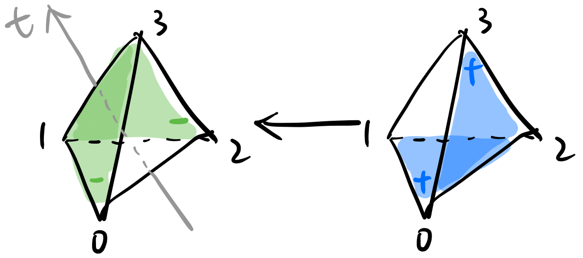
(space) F :

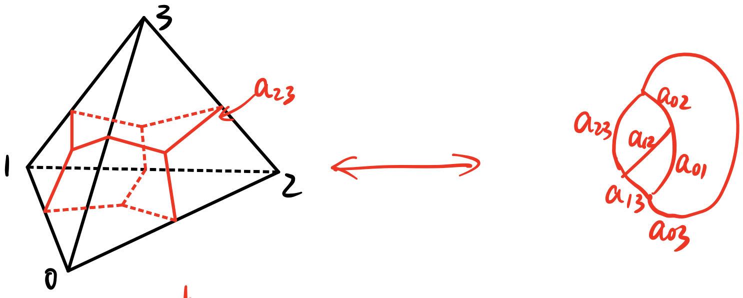


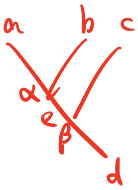
$V_{123} \otimes V_{013} \longleftarrow V_{012} \otimes V_{023}$

$$\left(F^{a_{23},a_{12},a_{01}}_{a_{03}}\right)_{a_{13},a_{02}} \sim \begin{pmatrix} a_{01} & a_{02} & a_{12} \\ a_{23} & a_{13} & a_{03} \end{pmatrix}$$ 6j-symbol

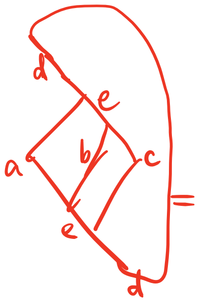
(Spacetime) F move corresponds to a tetrahedron in 2+1D spacetime.



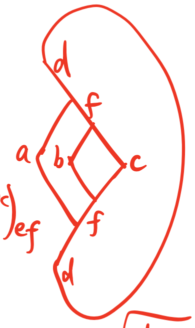




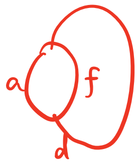
$= \sum_f \left(F^{abc}_d\right)_{ef}$



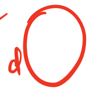
$= \sum_f \left(F^{abc}_d\right)_{ef}$



$= \sum_f \left(F^{abc}_d\right)_{ef} \sqrt{\frac{d_b d_c}{d_f}}$

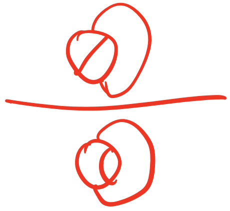


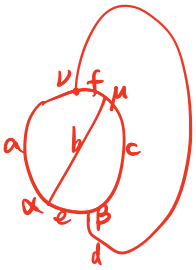
$= \sum_f \left(F^{abc}_d\right)_{ef} \sqrt{\frac{d_b d_c}{\cancel{d_f}}} \sqrt{\frac{d_a \cancel{d_f}}{d_d}}$



$= \sum_f \left(F^{abc}_d\right)_{ef} \sqrt{d_a d_b d_c d_d}$

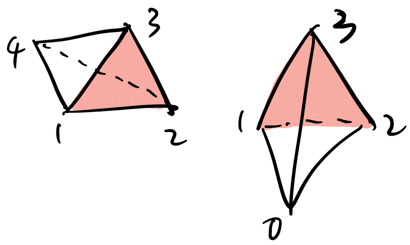
or $\left(F^{abc}_d\right)_{ef} =$



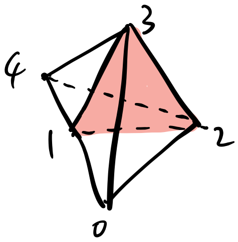


$$V_{012} \otimes V_{023} \otimes V^*_{013} \otimes V^*_{123}$$
$$\|$$
$$\mathrm{Hom}(a\otimes c, d) \otimes \mathrm{Hom}(a\otimes b, c) \otimes \mathrm{Hom}(f, b\otimes c) \otimes \mathrm{Hom}(d, a\otimes f) \to \mathbb{C}$$
$$\beta \otimes \alpha \otimes \mu \otimes \nu \mapsto \mathrm{Tr}\left[\beta(\alpha\otimes 1)(1\otimes\mu)\nu\right]$$

- glue tetrahedron



glue $\langle 0123\rangle$ and $\langle 1234\rangle$ along $\langle 123\rangle$



$$M_3 = \langle 0123\rangle \underset{\langle 123\rangle}{\cup} \langle 1234\rangle$$

$$\langle 0123\rangle : \quad V_{012}\otimes V_{023}\otimes V_{013}^{*} \otimes V_{123}^{*} \longrightarrow \mathbb{C}$$

$$\langle 1234\rangle : \quad V_{123}\otimes V_{134}\otimes V_{124}^{*} \otimes V_{234}^{*} \longrightarrow \mathbb{C}$$

$$\Longrightarrow \quad M_3 : \quad V_{012}\otimes V_{023}\otimes V_{013}^{*} \otimes V_{134}\otimes V_{124}^{*} \otimes V_{234}^{*} \longrightarrow \mathbb{C}$$

- general M_3 with boundary : $\partial M_3 \to \mathbb{C}$

  - - - without boundary : $\bigotimes_{\text{face } f} \left(V_f \otimes V_f^{*}\right) \longrightarrow \mathbb{C}$

$f = \partial t_1 = -\partial t_2$

for tetrahedra t_1, t_2

$$Z(M_3, l)$$

3-manifold with a triangulation

coloring l: $E \to \mathrm{obj}(\mathcal{C})$

$V \to \mathrm{Hom}_{\mathcal{C}}(-,-)$

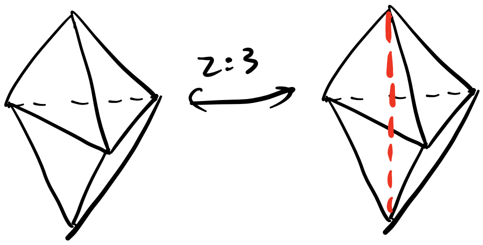
- TV invariants

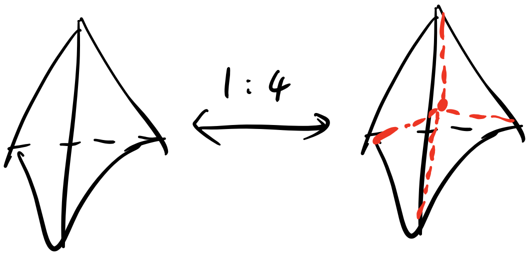
$$Z_{TV}(M_3) := (\dim \mathcal{C})^{-|V|} \sum_{l} Z(M_3, l) \prod_{e\in E} d_{l(e)}$$

vertices   coloring   edges   q-dim of $l(e)$

- Invariance of Z_{TV} under Pachner move.

  3D Pachner move :





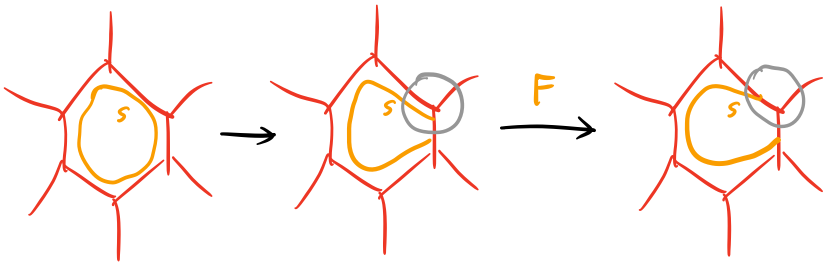
pentagon equation for F moves

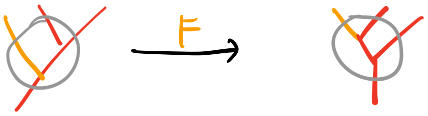
$\Rightarrow$ invariance of Z_{TV} under 3D Pachner move

$\Rightarrow$ Z_{TV} is a 3-manifold invariant

(2) Path integral of Levin-Wen model.

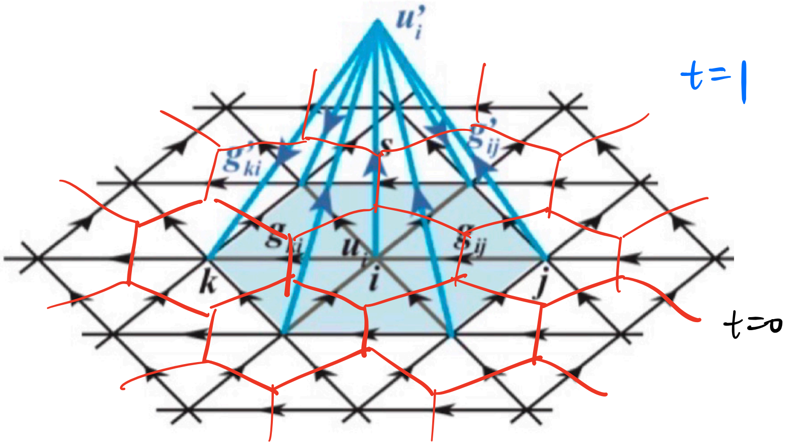
B_p^s :

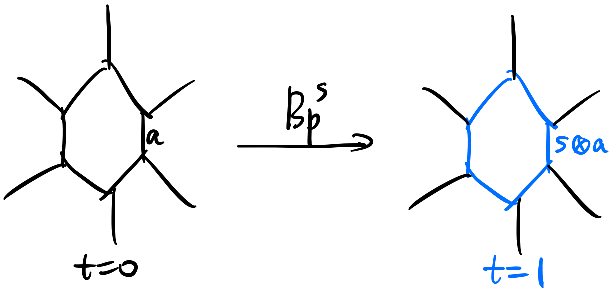




one tetrahedron for one F move.

B_p^s : 6 F moves $\rightarrow$ 6 tetrahedra.

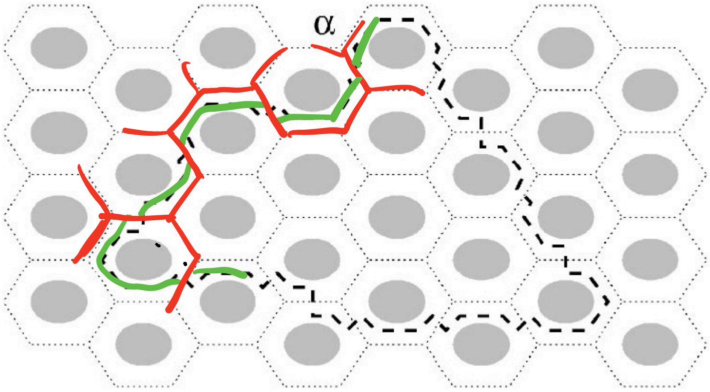


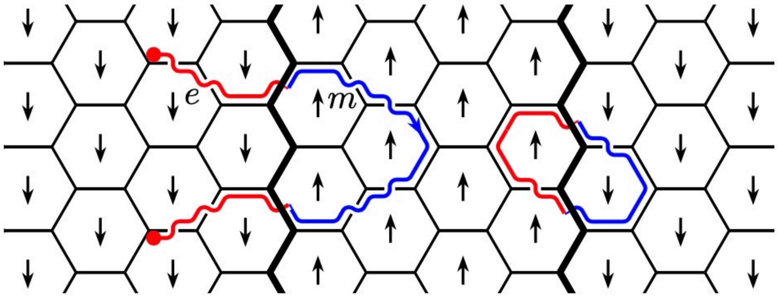


(3) String operators and excitations.

excitations : $A_s \neq 1$ or $B_p \neq 1$

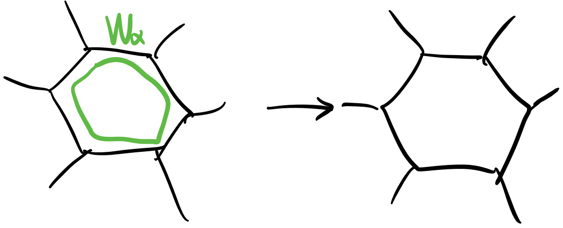
expect that excitations can be created at the endpoints of string operators.



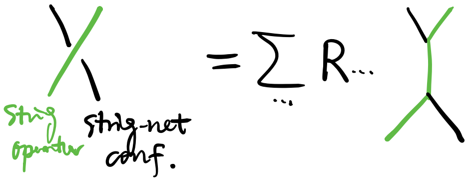


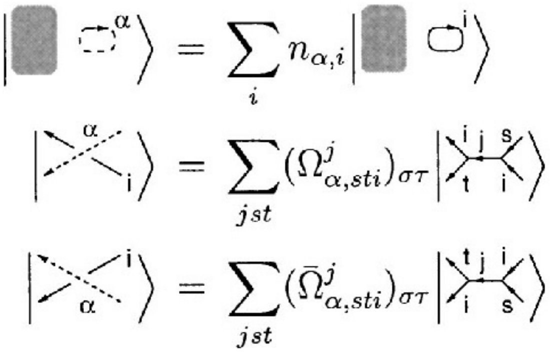
$$[W_\alpha , \begin{smallmatrix} A_s \\ B_p \end{smallmatrix}] = 0 \quad \text{for } s,p \notin \partial\alpha$$

- closed string operator fuses to ground state.

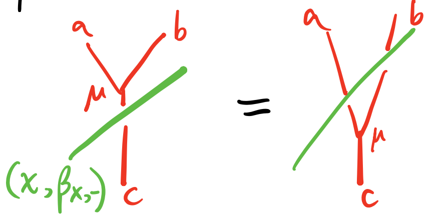


- string operator should be labelled by obj($\mathcal{C}$)
- string-net conf $\xrightarrow[\text{operator}]{\text{string}}$ string-net conf

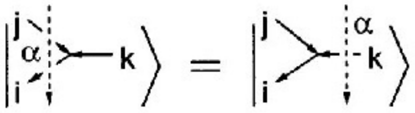


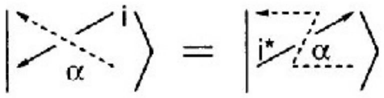


- string operator commute with A_s, B_p.



(naturality)





$\Longrightarrow$ excitations of Levin-Wen model are described by Drinfeld center $Z(\mathcal{C})$.