10. Introduction to fermionic SPT.

10.1. 1+1D TSC in class D

$$\underbrace{\text{class D}}_{(C^2=1,\ T=0,\ S=0)}$$

• BCS for spinless 1+1D p-wave SC :

$$\hat{H} = \sum_j \left[-t\left(c_j^\dagger c_{j+1} + h.c.\right) + \left(\Delta c_j c_{j+1} + \Delta^* c_{j+1}^\dagger c_j^\dagger\right)\right]$$

$$\begin{cases} c_j = \frac{1}{\sqrt{L}} \sum_k e^{-ikj} c_k \\ c_k = \frac{1}{\sqrt{L}} \sum_j e^{ikj} c_j \end{cases}$$

$$= \sum_k \left[-2t \cos k \, c_k^\dagger c_k + \left(-i\Delta \sin k \, c_{-k} c_k + h.c.\right)\right]$$

$$= \sum_k \left(c_k^\dagger, c_{-k}\right) \underbrace{\begin{pmatrix} -t\cos k & i\Delta^* \sin k \\ -i\Delta \sin k & +t\cos k \end{pmatrix}}_{\mathcal{H}_k} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} + \text{const.}$$

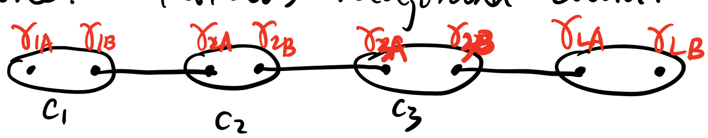
$\mathcal{H}_k$ has particle-hole symmetry:

$$\left.\begin{aligned} &\tau_x \mathcal{H}_{-k}^T \tau_x^{-1} = -\mathcal{H}_k \\ &(\tau_x)^2 = I \end{aligned}\right\} \Rightarrow C = \tau_x \qquad C^2 = 1.$$

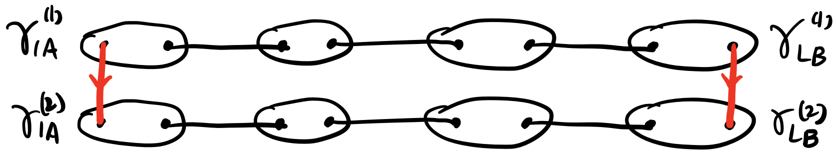
For interacting system, class D $\to$ $G_f = \mathbb{Z}_2^f$.

• Classification : $\mathbb{Z}_2$.

Nontrivial one: Kitaev's Majorana chain.



$$c_j = \frac{1}{2}\left(\gamma_{jA} + i\gamma_{jB}\right)$$



Add fermion bilinear form

$$-i\gamma_{1A}^{(1)}\gamma_{1A}^{(2)} - i\gamma_{LB}^{(1)}\gamma_{LB}^{(2)}$$

$$= (1-2a_1^\dagger a_1) + (1-2a_L^\dagger a_L)$$

$a_1 = \frac{1}{2}(\gamma_{1A}^{(1)} + i\gamma_{1A}^{(2)})$

$a_L = \ldots$

— $a^\dagger a_1 = 1$

— $a_1^\dagger a_1 = 0$

$\Rightarrow$ Two Majorana chain is trivial in class D.

$\Rightarrow$ $\mathbb{Z}_2$ classification for $\begin{cases}\text{noninteracting} \\ \text{interacting}\end{cases}$ class D TSC.

10.2. 1+1D TSC in $\underline{\text{class BDI}}$.

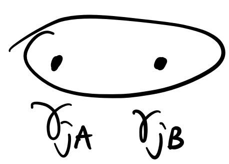
$\downarrow$

$C^2=1,\ T^2=1,\ S=1$

$$T: \begin{cases} c_j \longrightarrow c_j \\ i \longrightarrow -i \end{cases}$$

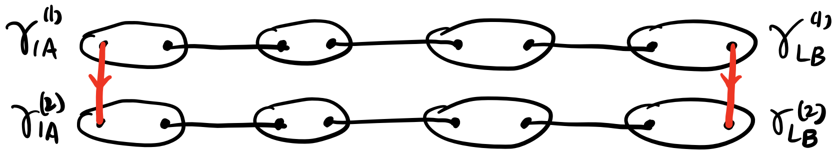
$$c_j = \frac{1}{2}(\gamma_{jA} + i\gamma_{jB})$$

$$\Rightarrow T: \begin{cases} \gamma_{jA} \longrightarrow \gamma_{jA} \\ \gamma_{jB} \longrightarrow -\gamma_{jB} \end{cases}$$



Majorana chain:

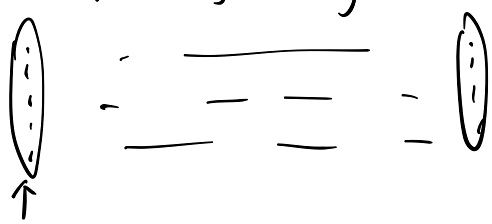
$$\hat{H} = \sum_j -i\gamma_{j,B}\gamma_{j+1,A} \xrightarrow{T} \hat{H}$$



Add fermion bilinear form

$$H_{bdy} = -i\,\gamma_{1A}^{(1)}\gamma_{1A}^{(2)} - i\,\gamma_{LB}^{(1)}\gamma_{LB}^{(2)} \xrightarrow{T} -H_{bdy}$$

Consider N copies of Majorana chains :



$\{\gamma_{1A}^{(1)}, \gamma_{1A}^{(2)}, \cdots, \gamma_{1A}^{(N)}\}$

free fermion : $H_{bdy} = \sum_{j,k} -i J_{jk}\gamma_{1A}^{(j)}\gamma_{1A}^{(k)}$

$$\Big\downarrow T$$

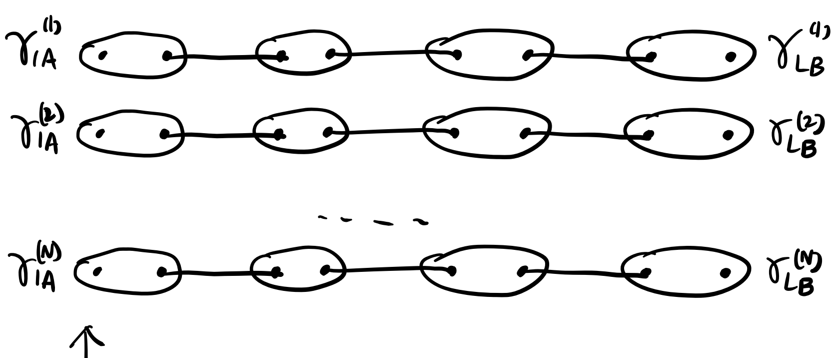
$$-H_{bdy}$$

$\Rightarrow$ free fermion classification for 1+1D class BDI TSC is $\mathbb{Z}$.

10.3. Interaction effect of 1+1D TSC in class BDI.

(Fidkowski, Kitaev 2009)

noninteracting 1+1D BDI classifield by $\mathbb{Z}$ $\xrightarrow[\text{interaction}]{\text{add}}$ fermionic SPT with $G_f = \mathbb{Z}_2^f \times \mathbb{Z}_2^T$ classified by $\mathbb{Z}_8 = \mathbb{Z}/8\mathbb{Z}$



$\{\gamma_{1A}^{(1)}, \dots, \gamma_{1A}^{(N)}\} =: \{\gamma_1, \dots, \gamma_N\}$

Criteria: trivial SPT $\Longleftrightarrow$ $\exists$ trivial edge state

(symmetric, GSD $=1$, no SSB)

All possible interaction terms of one edge:

$$\begin{cases} \cancel{i\gamma_j\gamma_k} & (j<k) & \to \text{2-body} \\ \gamma_j\gamma_k\gamma_l\gamma_m & (j<k<l<m) & \to \text{4-body} \\ \cancel{i\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\gamma_{j_5}\gamma_{j_6}} & (j_1<\dots<j_6) & \to \text{6-body} \\ \dots \end{cases}$$

$\downarrow$ time reversal symmetric

$$\begin{cases} \text{4-body} \\ \text{8-body} \\ \dots \end{cases}$$

$n=1$

edge state:

a Majorana fermion

① $n=2$. $\{\gamma_1, \gamma_2\}$

No symmetric edge term.

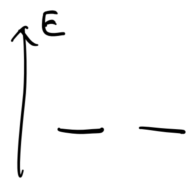
edge state: $-\gamma_1\gamma_2 = \pm 1$

bosonic / fermionic

GSD $=2$.

edge state:

a complex fermion

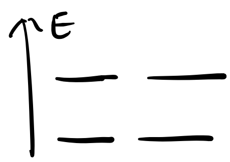


② $n=4$. $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$

$$H_{edge} = \gamma_1\gamma_2\gamma_3\gamma_4 = -\underbrace{(-i\gamma_1\gamma_2)}_{P_f}\underbrace{(-i\gamma_3\gamma_4)}_{P_f}$$

$|GS\rangle = |00\rangle$ or $|11\rangle$

$n_1 = c_1^\dagger c_1$, $n_2 = c_2^\dagger c_2$



$c_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$

$c_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)$

edge state:

$|\uparrow\rangle, |\downarrow\rangle$

$|\uparrow\rangle = |00\rangle$, $|\downarrow\rangle = |11\rangle$

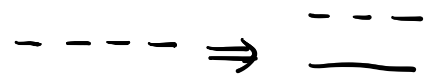
Haldane chain

③ $n=8$. $\{\gamma_1, \dots, \gamma_8\}$

$$H_{edge} = \gamma_1\gamma_2\gamma_3\gamma_4 + \gamma_5\gamma_6\gamma_7\gamma_8$$

$$GS = \left(|0\,0\rangle_{1234} \uparrow \text{ or } |1\,1\rangle_{1234} \downarrow \right) \otimes \left(|0\,0\rangle_{5678} \uparrow \text{ or } |1\,1\rangle_{5678} \downarrow \right)$$

Hilbert space: spin $\frac{1}{2}$ $\otimes$ spin $\frac{1}{2}$



$$|GS\rangle = \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) = |0011\rangle - |1100\rangle$$

n_1 n_2 n_3 n_4

n_i for 1,2

$a_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$

$$H_{edge} = \gamma_1\gamma_2\gamma_3\gamma_4 + \gamma_5\gamma_6\gamma_7\gamma_8 + \vec{S}_{1234} \cdot \vec{S}_{5678}$$

$\uparrow \Leftrightarrow 00$ n_j

$\downarrow \Leftrightarrow 11$

$c_{ij} = \frac{1}{2}(\gamma_i + i\gamma_j)$

$$\begin{cases} S^-_{1234} = |\downarrow\rangle\langle\uparrow| = |11\rangle\langle 00| = c^\dagger_{12}\, c^\dagger_{34} \\ S^+_{1234} = c_{34}\, c_{12} \\ S^z_{1234} = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| \right) \end{cases}$$

n_i

$$= \frac{1}{2}\left(|0\,0\rangle\langle 0\,0| - |1\,1\rangle\langle 1\,1| \right)$$

$$= \frac{1}{2}\left[(1-n_{12})(1-n_{34}) - n_{12}\, n_{34} \right]$$

$$H_{AF} = \vec{S}_{1234} \cdot \vec{S}_{5678}$$

$$= \frac{1}{2}\left(S^+_{1234} S^-_{5678} + h.c. \right) + S^z_{1234} S^z_{5678}$$

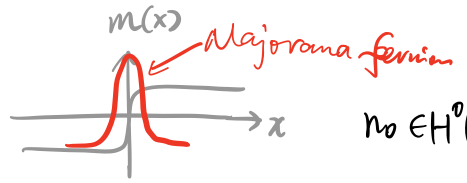
$$= \dots \text{ (Majorana fermion terms)}$$

$\Rightarrow$ 8 copies of Majorana chains is a trivial FSPT with $G_f = \mathbb{Z}_2^f \times \mathbb{Z}_2^T$.

10.4. Group supercohomology theory of FSPT.

1+1 D.

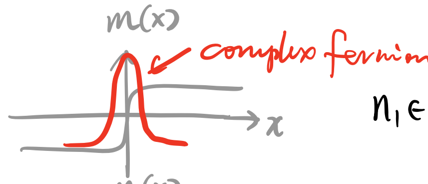
Kitaev chain, edge Majorana fermion



$n_0 \in H^0(G, \mathbb{Z}_2)$

$\downarrow \times 2$ 1/2

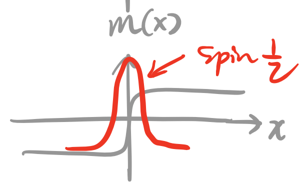
FSPT with edge complex fermion



$n_1 \in H^1(G, \mathbb{Z}_2)$

$\downarrow \times 2$ 1/2

BSPT (Haldane chain) with edge spin $\frac{1}{2}$

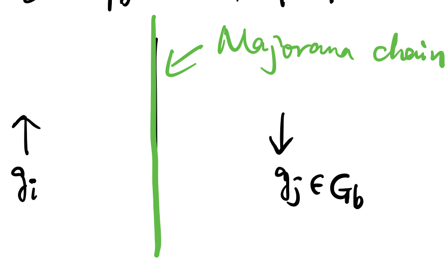


$\nu_2 \in H^2(G, U(1))$

$\downarrow \times 2$ 1/2

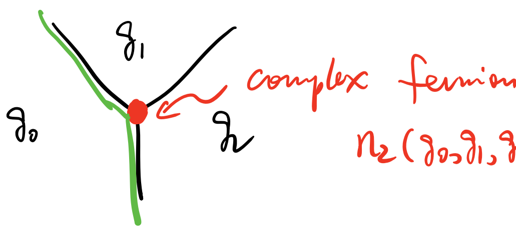
trivial

2+1 D. $G_f = \mathbb{Z}_2^f \times G_b$ FSPT.



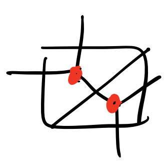
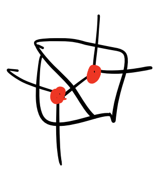
$n_1(g_i, g_j) \in H^1(G_b, \mathbb{Z}_2)$

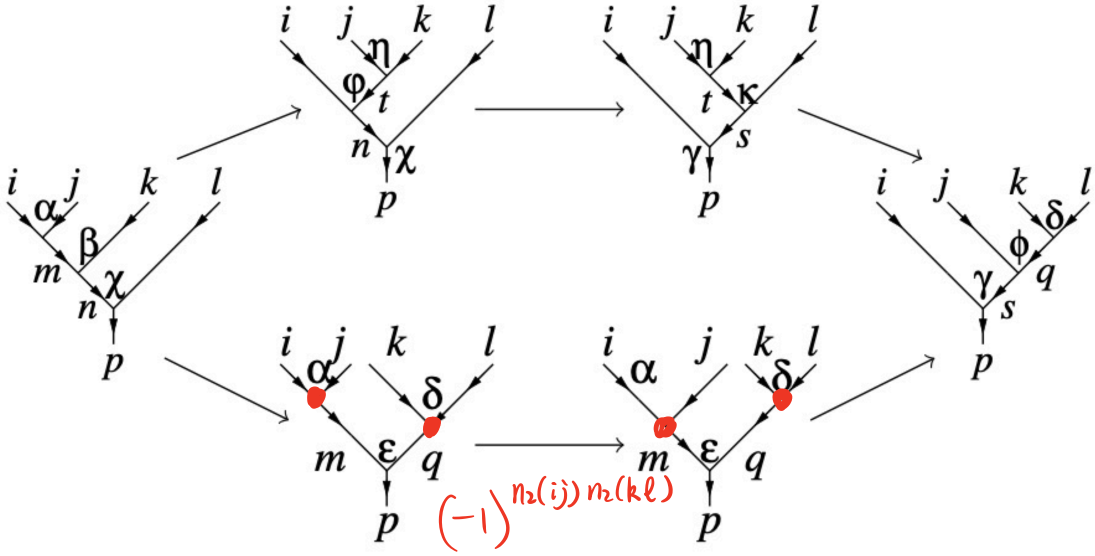
$(dn_1)(g_0, g_1, g_2)$
$= n_1(g_1, g_2) + n_1(g_0, g_2) + n_1(g_0, g_1)$
$= 0 \pmod 2$
closed Majorana chain.



$n_2(g_0, g_1, g_2)$

$(dn_2)(g_0, g_1, g_2, g_3)$
$= n_2(\cdots) - n_2(\cdots) + n_2(\cdots) - n_2(\cdots)$
$= 0 \pmod 2$
fermion parity conservation

 $\longleftarrow \nu_3$ 



BSPT: $d\nu_3 = 1$

FSPT: $d\nu_3 = (-1)^{n_2 \cup n_2}$

$$\begin{cases} n_1 \in H^1(G_b, \mathbb{Z}_2) \\ n_2 \in H^2(G_b, \mathbb{Z}_2) \\ \nu_3 \in C^2(G_b\, \mathbb{Z}_2) \end{cases}$$

Majorana chain decoration

complex fermion –

BSPT

$$\begin{cases} dn_1 = 0 \pmod 2 \\ dn_2 = 0 \pmod 2 \\ d\nu_3 = (-1)^{n_2 \cup n_2} \end{cases}$$

closed chain

P_f conservation

super pentagon