Topological Quantum Matters (TQM)

Time:

Every Mon. & Wed. 13:30-15:05, from 2021-9-13 to 12-3 (weeks 1-12 of the fall semester in Tsinghua)

Venue:

It will be a combination of offline (宁斋W11, Ning Zhai W11) and online (腾讯会议tencent meeting: 5772849861, password: 654321)

Description:

In this course, we will use topology to understand some exotic quantum phases of matter. The topics will include topological insulators, topological orders, symmetry-protected topological phases, etc. The course will cover both condensed matter models in physics and general mathematical descriptions (such as group cohomology theory and modular tensor categories of knots).

Prerequisites:

Basic topology and quantum mechanics. We will try to make a compromise between mathematics and physics by introducing relevant concepts self-consistently, as there are audience from both sides.

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other courses:

YMSC
- Reshetikhin : Inv. of knots and 3-mfd.
- Haghighat : Topological Quantum Computation.
- Zheng : Category theory

BIMSA
- Papadimitriou : Quantum field theory anomalies and applications
- Palcoux : On fusion categories
- Yilong Wang : Modular categories and Reshetikhin-Turaev TQFTs

Introduction to TQM.

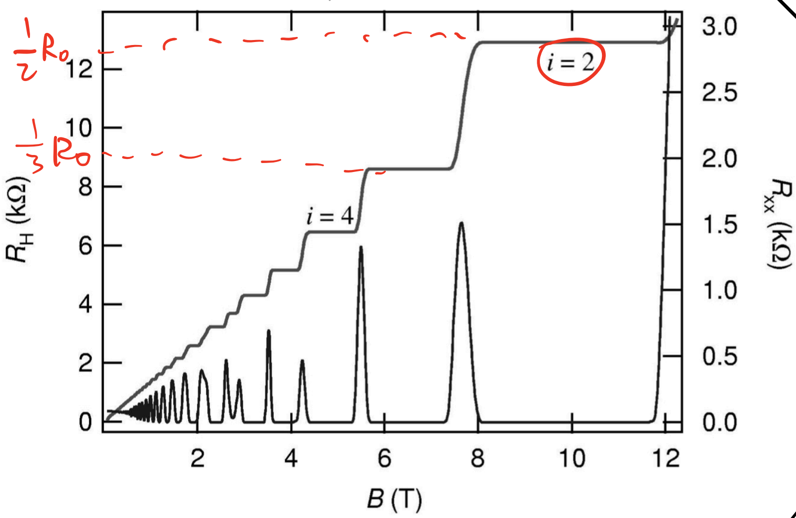
(1) Why important to real life?

One example:

Q: how to measure $\begin{cases} \text{electric charge } e \\ \text{Planck constant } h \end{cases}$ accurately?

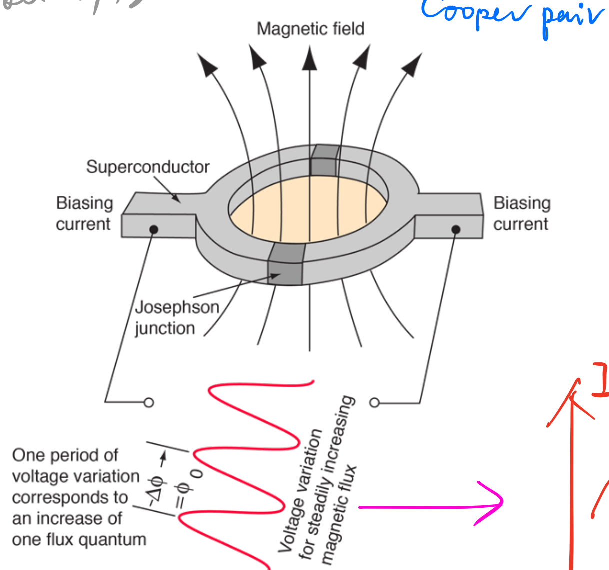
① Integer quantum Hall effect $\qquad R_0 = \dfrac{h}{e^2}$

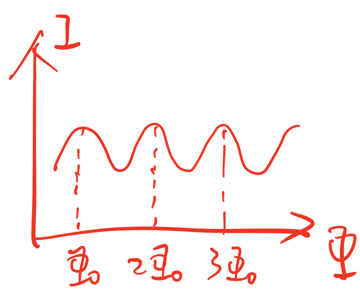
Nobel 1985



② Josephson effect $\qquad \Phi_0 = \dfrac{h}{2e}$ Cooper pair

Nobel 1973





$$\begin{cases} e = \dfrac{2\Phi_0}{R_0} \\ h = \dfrac{4\Phi_0^2}{R_0} \end{cases}$$

$\Downarrow$

International System of Units (before 2018)

[After 2018, e, h are fixed by definition]

using many-body system to measure accurately properties of a single electron!

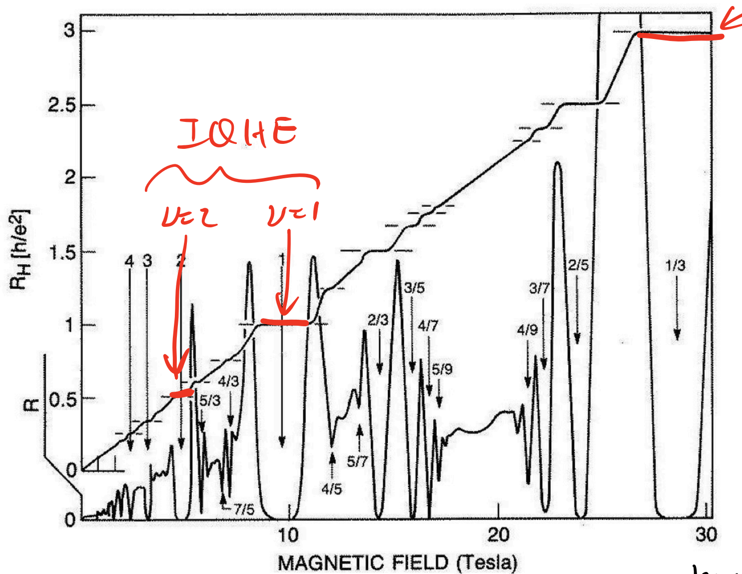
(2) Why exotic?

fundamental charge e = charge of one single electron/proton/...

Fractional quantum Hall effect (FQHE) Nobel 1998

$$R = \frac{h}{\nu e^2} \qquad \begin{cases} \nu \in \mathbb{Z}: & \text{integer QHE} \\ \nu \in \mathbb{Q}: & \text{FQHE} \end{cases}$$

Fractional charge $q = \nu \cdot e$



$\nu = \frac{1}{3}$ Laughlin state
$(q = \frac{1}{3} e)$

Fractional charge appears in many-body systems!

(3) Why interesting to many people?

related to many other topics:

- math: knot theory, Jones polynomial, Turaev-Viro inv, Reshetikhin-Turaev inv., modular tensor categories, Zheng's lecture
  lecture 2-knot, higher categories ...
- physics: IQHE, FQHE, topological insulator, ...
  (real materials)
- quantum information: topological quantum computation
  (by Freedman, Kitaev)
  fault-tolerant quantum computation

(4) Why hard/deep?

∞ number of degrees of freedom

∞ dimensional space.

fundations of quantum field theory (= many-body quantum mechanics)

emergence of spacetime/gravity (?)

Syllabus (tentative):

It may vary depending on the actual speed of the course. By weeks (4*45min/week, 12 weeks):

(1) introduction to TQM, different classes of TQM (bosonic/fermionic, long/short range entangled, with/without symmetry),
1st example: Kitaev's toric code model (homology enters),
2nd example: Haldane's honeycomb model (homotopy enters) ⇝ Qikun Xue : quantum anomalous Hall effect.
1988 2013

Part I. Bosonic topological orders

(2) quantum double model, twisted quantum double model = Dijkgraaf-Witten gauge theory

(3) introduction to fusion categories, Levin-Wen model = Turaev-Viro model

(4) introduction to modular tensor categories, general description of anyon models by Kitaev

(5) 3+1D Walker-Wang model = Crane-Yatter model

Part II. Topological insulators (fermionic symmetry-protected topological phases without interactions)

(6) introduction to band theory, integer quantum Hall effect, Thouless–Kohmoto–Nightingale–den Nijs number, Chern insulator
TKNN : Hall conductance = Chern number

(7) examples: Kitaev's Majorana chain, Su-Schrieffer-Heeger model, 2+1D and 3+1D topological insulators, edge theories

(8) symmetries in free fermion system, 10-fold way classification → topological K theory, Clifford algebra

Part III. Symmetry-protected topological phases

(9) introduction to symmetry-protected topological (SPT) phases, Haldane chain

(10) introduction to projective representation, tensor product state, classification of 1+1D bosonic SPT

(11) Levin-Gu model, introduction to group cohomology, bosonic SPT model from group cohomology

(12) introduction to fermionic SPT phases, other related topics

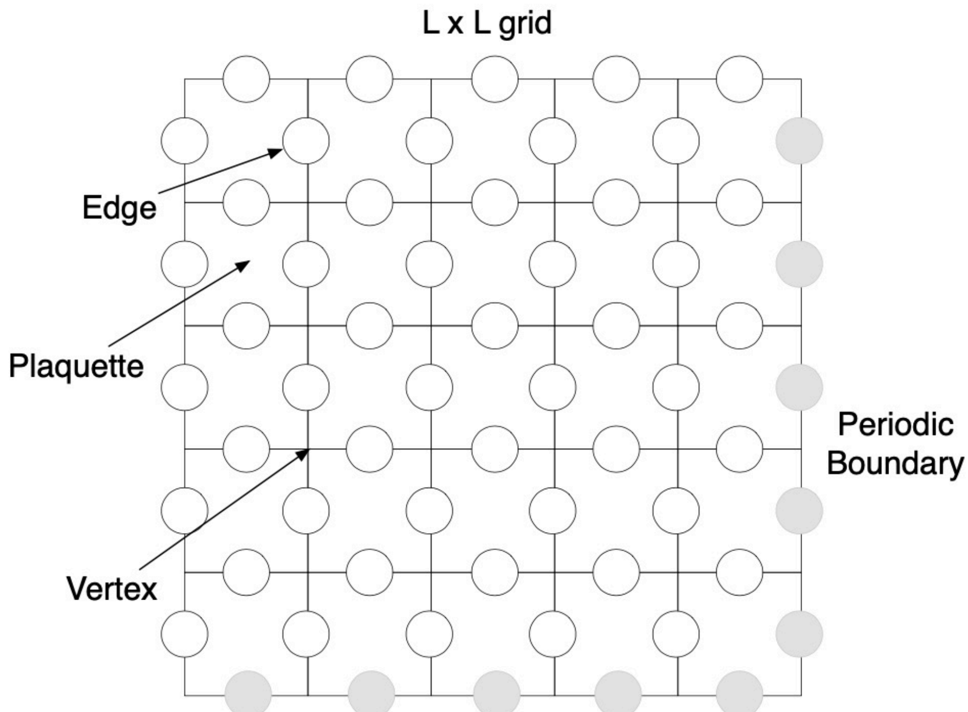
1st example of TQM:

Toric code model (by Kitaev 1997) $\left(= \mathbb{Z}_2 \text{ gauge theory on torus}\right)$

Hilbert space on torus

$L \times L$ square lattice, $\mathbb{C}^2$ (spin 1/2) on each link

$$\mathscr{H} = \otimes_{\text{link } j} \mathbb{C}^2$$



$\mathcal{H}$: Hilbert space

Hamiltonian $H : \mathcal{H} \to \mathcal{H}$

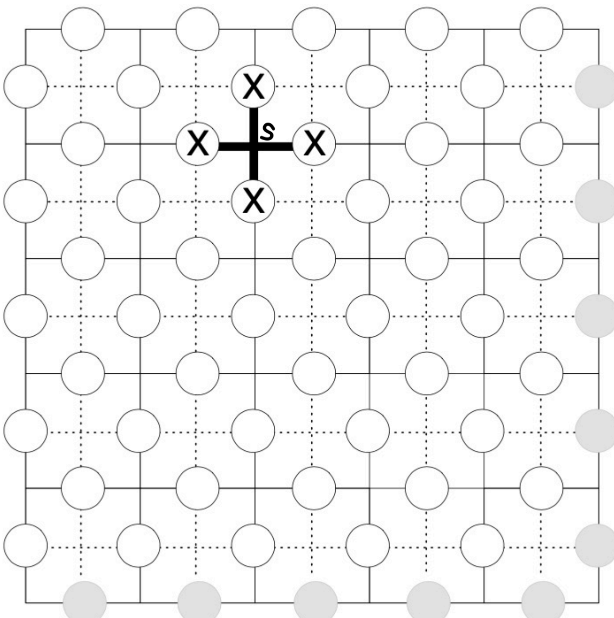
eigenvalues of H are energies ε_n of the system

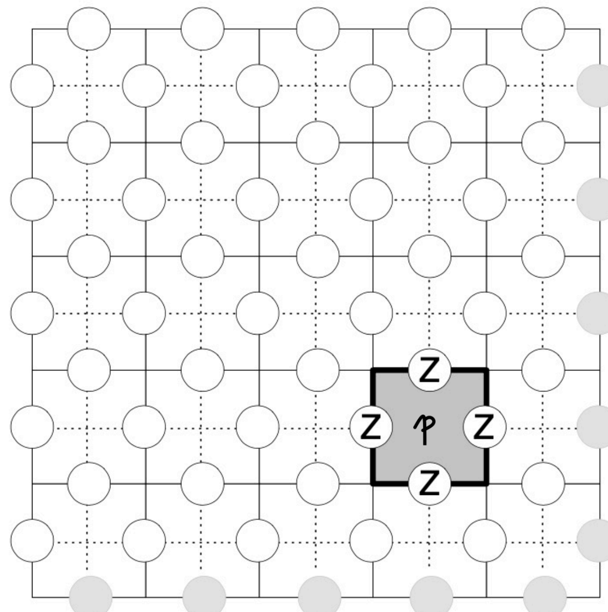
Ground state is the eigenvector with the lowest energy ε_0.

Claim
- The ground state of TC represents $H_1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$
- robust ground states
- Topological quantum information

Hamiltonian

$$H = -\sum_{\text{site } s} A_s - \sum_{\text{plaquette } p} B_p$$





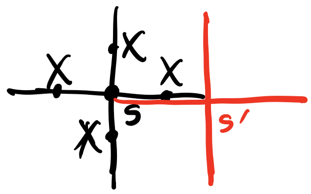
$$A_s = \prod_{j \in \text{star}(s)} \sigma_j^x$$ Pauli operator

$$X = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

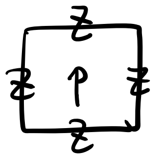
$$B_p = \prod_{j \in \partial p} \sigma_j^z$$

$$Z = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Algebraic relations of A_s / B_p.

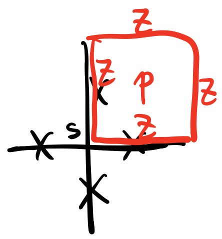


$$\begin{cases} A_s^{+} = A_s \\ (A_s)^2 = 1 \\ [A_s, A_{s'}] = A_s A_{s'} - A_{s'} A_s = 0 \end{cases}$$ $\rightsquigarrow$ A_s has eigenvalues ± 1



$$\begin{cases} B_p^{+} = B_p \\ (B_p)^2 = 1 \\ [B_p, B_{p'}] = 0 \end{cases}$$ $\rightsquigarrow$ B_p $\cdots$ ± 1.

$$[A_s, B_p] = A_s B_p - B_p A_s = 0.$$



$XZ = -ZX \Rightarrow A_s B_p = B_p A_s$

$$H = -\sum_s A_s - \sum_p B_p$$

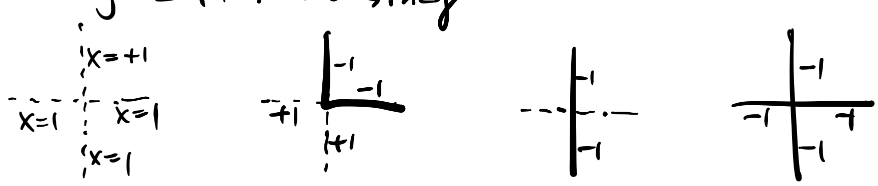
Ground state $|\Psi\rangle$ of TC :

$$A_s |\Psi\rangle = B_p |\Psi\rangle = +1\, |\Psi\rangle, \quad \forall s, p.$$

String representation. $\quad A_s = \prod \sigma^x. \qquad \sigma^x : \pm 1$

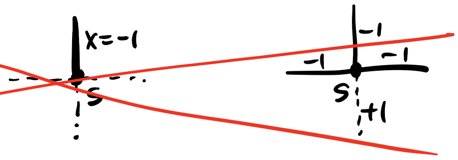
def: $\begin{cases} \sigma_j^x = -1 : \text{ string} \quad \text{———} \\ \sigma_j^x = +1 : \text{ no string} \quad \cdots\cdots \end{cases}$

low energy: $A_s = 1$:

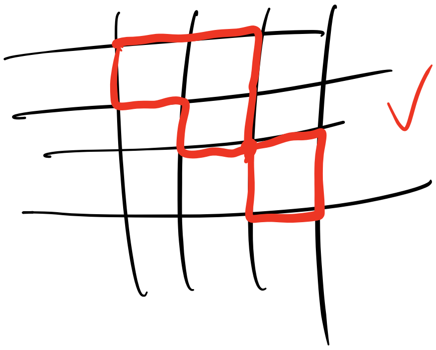


✓

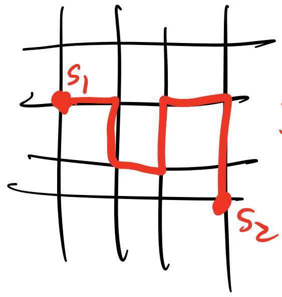
high energy: $A_s = -1$:



ground state : $A_s|\Psi\rangle = |\Psi\rangle \Longleftrightarrow$ closed strings (loops)

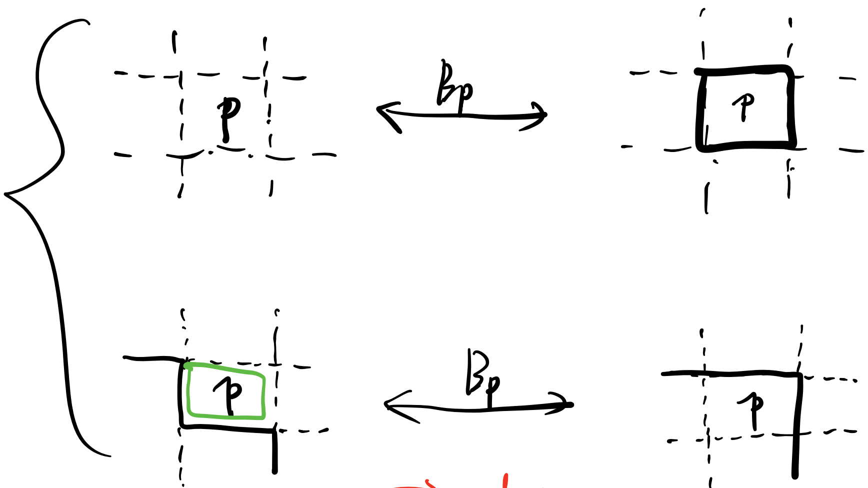


✓ $A_s = +1$ ($\forall s$)



✗ $A_{s_1} = A_{s_2} = -1$ (high energy)

$B_p = \prod Z$: flip $\sigma^x = +1 \Leftrightarrow \sigma^x = -1$



create annihilation small loops.

change shapes of closed loops.

$B_p |c\rangle = |c + \partial p\rangle$ mod 2

$\forall$ closed loop configuration

$H = -\sum A_s - \sum B_p$

closed loop. change shapes.

Assume $|\Psi\rangle = \sum_{\substack{\text{closed loop} \\ c}} a_c |c\rangle$, $a_c \in \mathbb{C}$ is the ground state

$\Rightarrow B_p |\Psi\rangle = \sum_c a_c |c + \partial p\rangle = \sum_c a_{c+\partial p} |c\rangle$

$= |\Psi\rangle = \sum_c a_c |c\rangle$

$\Rightarrow a_c = a_{c+\partial p}$ for $\forall p$

The ground state coefficients of $|c\rangle$ and $|c+\partial p\rangle$ should be the same.

or Loops that are homologous to each other have the same coefficient.

enforce closed-loop constraints

Summary
$\boxed{A_s = +1}$: conf. $|c\rangle$ such that there are only closed loops in c.

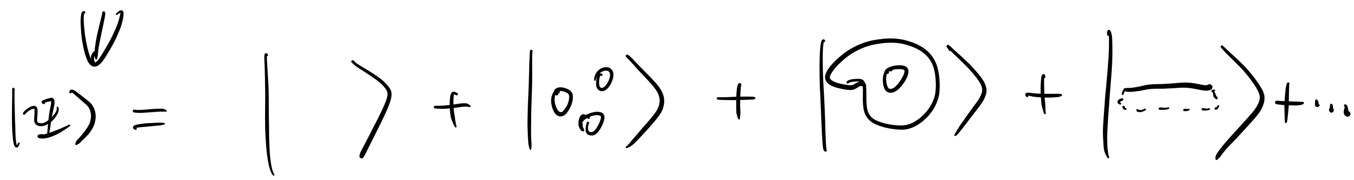
$\boxed{B_p = +1}$: $|c\rangle$ and $|c + \partial p\rangle$ (mod 2) have the same coefficient.

mod out homologous loops.

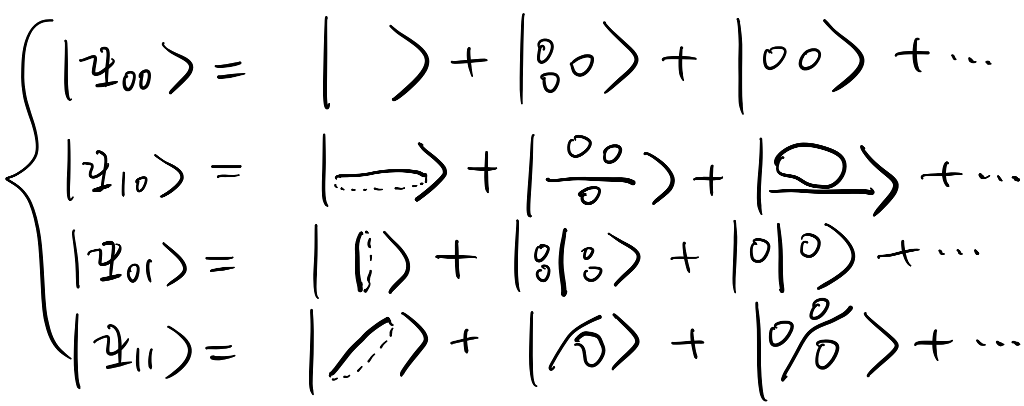
Ground state subspace of TC

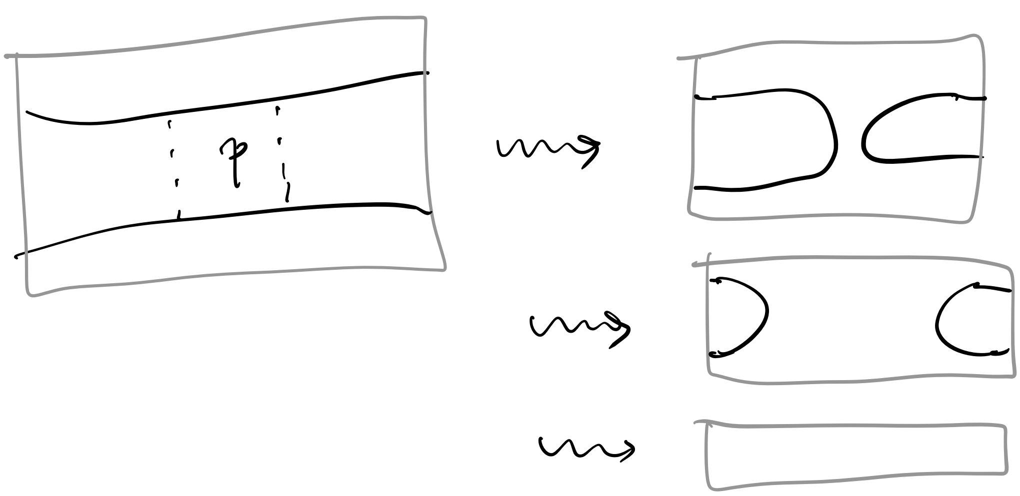
$= \{\text{closed loops}\} \,/\, \{\text{boundaries of 2D surfaces on torus}\}$

$= H_1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$



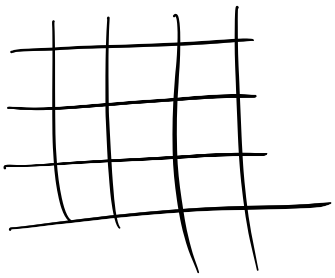
4-fold degeneracy of TC on torus :





Last time:

TC:



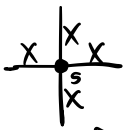
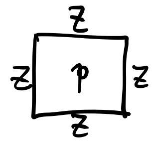
torus

square lattice

$$H = -\sum_{\text{site } s} A_s - \sum_{\text{plaquette } p} B_p$$

$(A_s)^2 = 1$

$\left(\frac{1+A_s}{2}\right)^2 = \underbrace{\frac{1+A_s}{2}}_{\text{projector to } A_s=1 \text{ states}}$

$A_s =$  $\qquad B_p =$ 

$[A_s, B_p] = 0$

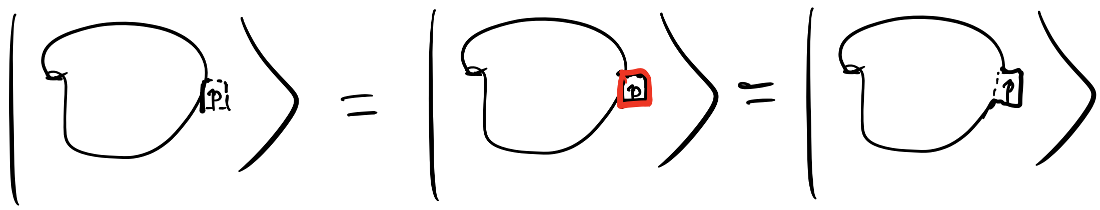
commuting-projector Hamiltonian.

$A_s = +1$ for every site s of the torus.

$\hookrightarrow$ closed string property of the GS.

($\sigma_j^x = -1$ for link j)

$B_p = +1$:

B_p 

GS of TC: $\quad A_s|\Psi\rangle = B_p|\Psi\rangle = |\Psi\rangle$, $\forall s, p$.

$$|\Psi\rangle = \sum_c a_c |c\rangle, \quad a_c \in \mathbb{C}$$

$$\begin{cases} A_s|\Psi\rangle = |\Psi\rangle \Rightarrow c \text{ is closed loop conf.} \\ B_p|\Psi\rangle = |\Psi\rangle \Rightarrow B_p \sum_c a_c|c\rangle = \sum_c a_c \underbrace{|c+\partial p\rangle}_{(\text{mod } 2)} \end{cases}$$

$$= \sum_c a_{c+\partial p} |c\rangle$$

$$= \sum_c a_c |c\rangle$$

$$\Rightarrow a_{c+\partial p} = a_c, \quad \forall p$$

$$\Rightarrow a : \mathcal{H}\big|_{A_s=1} \to \mathbb{C}$$
$$c \mapsto a_c$$

is a function of $\mathbb{Z}_2$ homology class of c.

Different classes of TQM:

① bosonic / fermionic

{ Microscopic degrees of freedom.
{ Not emergent excitations.

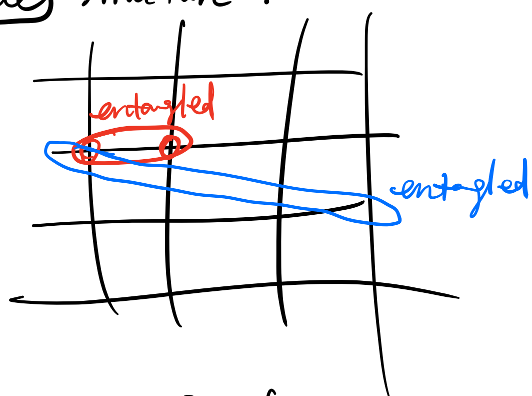
Example : bosonic Toric code has emergent fermions.

② long range enetangled / short range entangled
(LRE) (SRE)

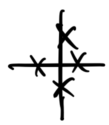
entanglement (in QM) $|\Psi\rangle = |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$

1st 2nd particle

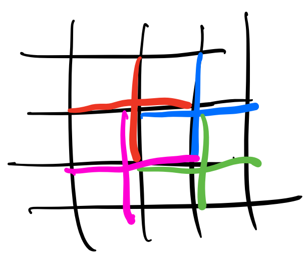
Add distance structure :



Example : T.C. is long range entangled.

$A_s =$ 

$\boxed{\prod_s A_s = 1.}$ on torus



$A_s = -1 \longrightarrow A_{s'} = -1$ for $s' \in \Lambda$

measure LRE : topological entanglement entropy.

$$S = (\text{area law}) + S_{top.} + \frac{\text{const.}}{L} + \cdots$$
$$\quad L \qquad\quad L^0 \qquad L^{-1}$$

③ with/without symmetries

Ising $\mathbb{Z}_2$ symm.

$SU(2)$, $U(1)$

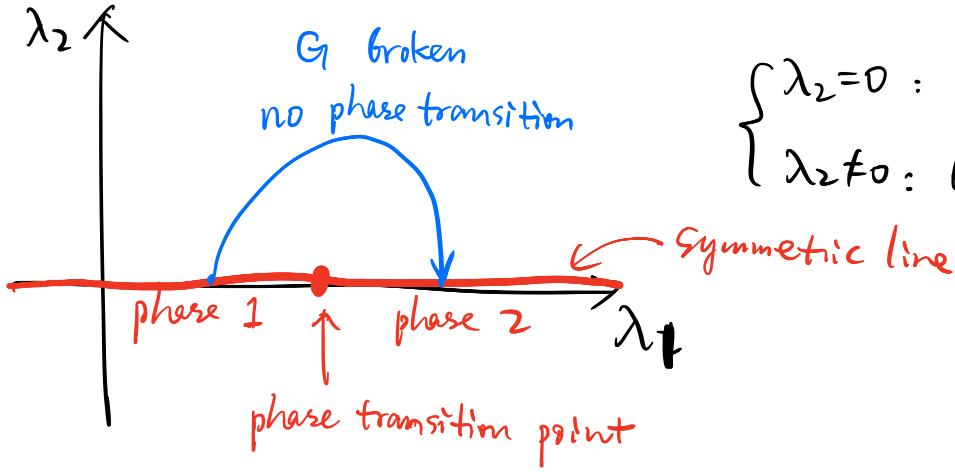
time reversal.

...

Given a symmetry group G, there may be many diffent topological phases. If we can break G, they may become the same phase.

G	3	4
phase 1	...	
2		

phase diagram

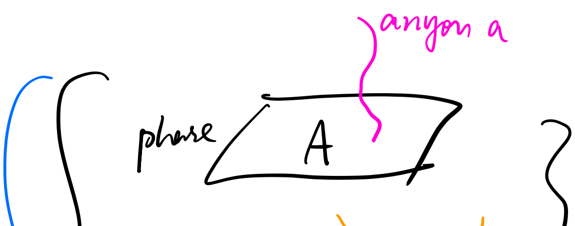


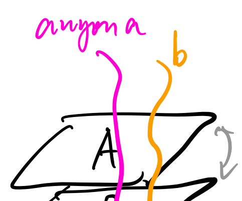
$$\begin{cases} \lambda_2 = 0 : & \text{symmetic} \\ \lambda_2 \neq 0 : & \text{break symm.} \end{cases}$$

① bosonic / fermion

② \ ③	SRE	LRE
No symm.	invertible topo. order (iTO)	topo. order (TO)
Symm.	Symmetry-protected topo. order (SPT)	symmetry-enriched topo. order (SET)

Statement: ① The collection of all topological phases forms a Abelian monoid (= group without inverse)
② The collection of all SRE phases has an Abelian group structure.





$A * B = B * A$

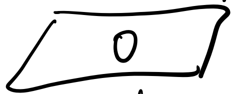
For topo. phases. (monoid)

phase 

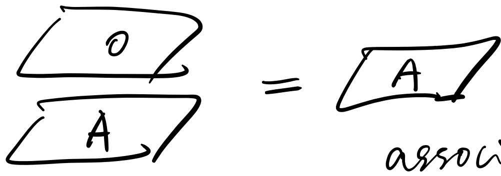
Stacking operation

For SRE phases (group)

trivial phase



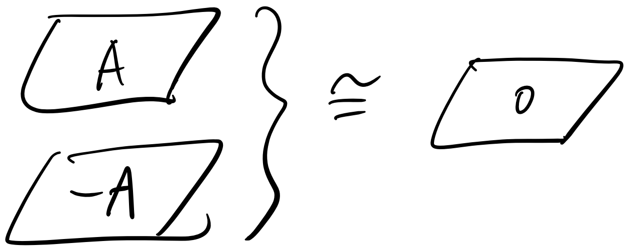
= product state



identity

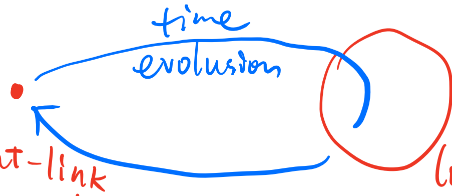
associativity obvious.

For SRE phases

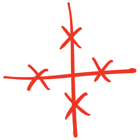


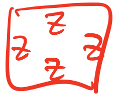
∃ inverse for ∀ SRE phase

3+1D TC:



$H = -\sum_s A_s - \sum_p B_p$

$A_s =$ 

$B_p =$ 

Wilson operators

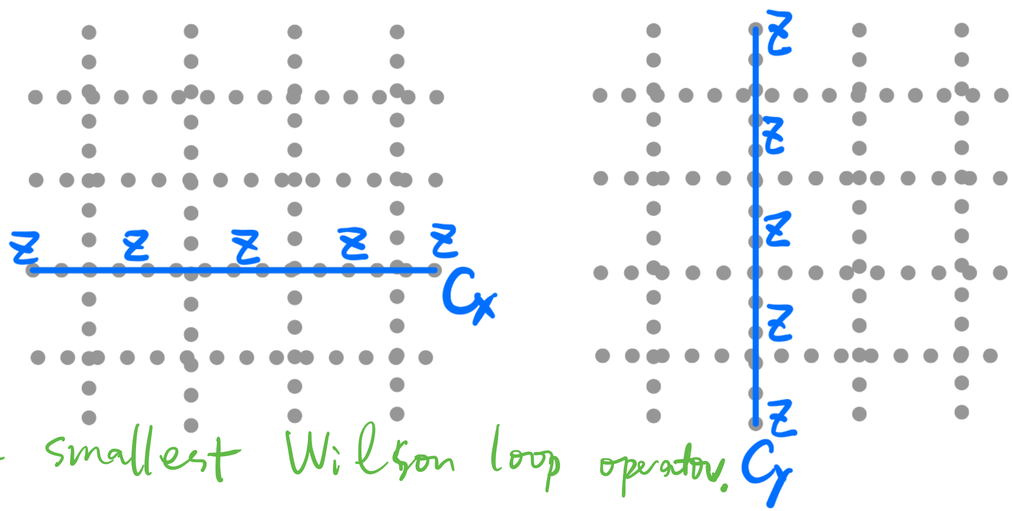
U(1) gauge theory

$W = e^{i \int_C A_\mu dx^\mu}$

For a loop C, define Wilson loop operator

$W_C = \prod_{j \in C} \sigma_j^z$, the ground states are related by

$$W_{C_x}, W_{C_y}$$

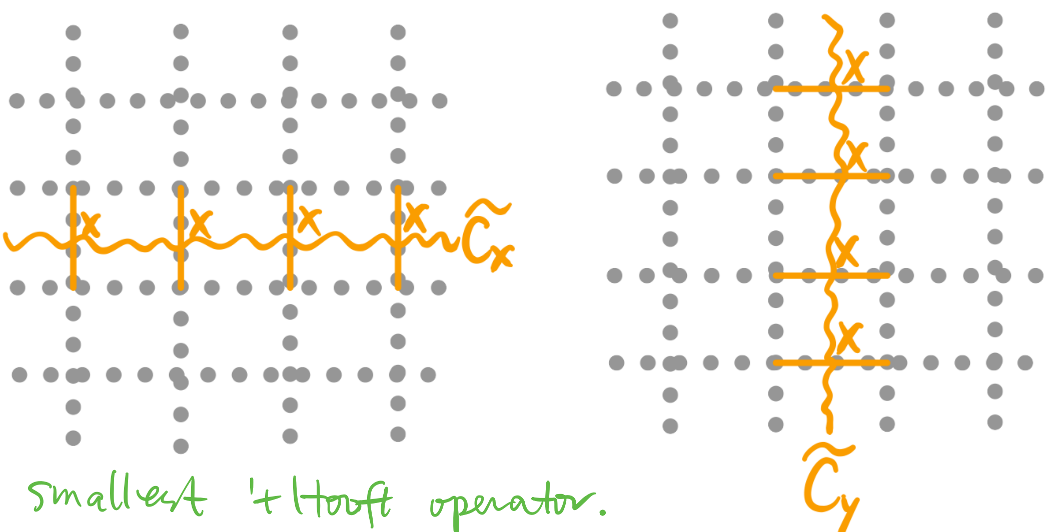


B_p is the smallest Wilson loop operator.

’t Hooft operators

For a dual loop C, define ’t Hooft operator $V_{\tilde{C}} = \prod_{j \perp \tilde{C}} \sigma_j^x$, the ground states are related by

$$V_{\tilde{C}_x}, V_{\tilde{C}_y}$$



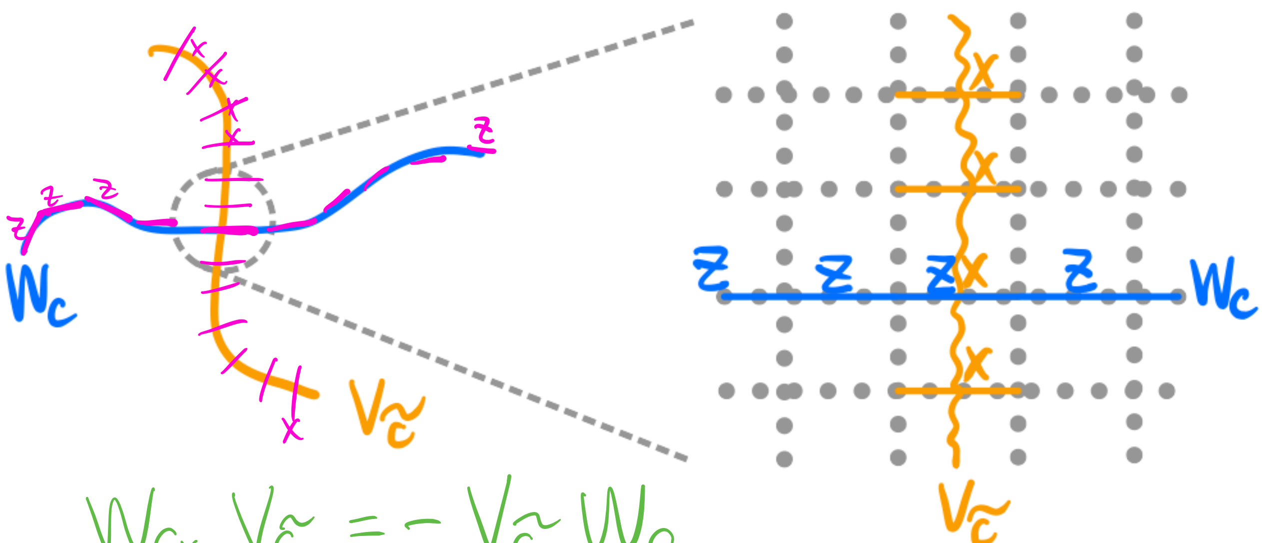
As is the smallest 't Hooft operator.

Heisenberg algebra from W and V

For a loop C and dual loop $\tilde{C}$, we have

$$W_C V_{\tilde{C}} = (-1)^{\# C \cap \tilde{C}} V_{\tilde{C}} W_C$$

which has smallest representation dim 4.

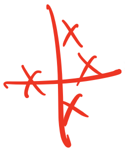


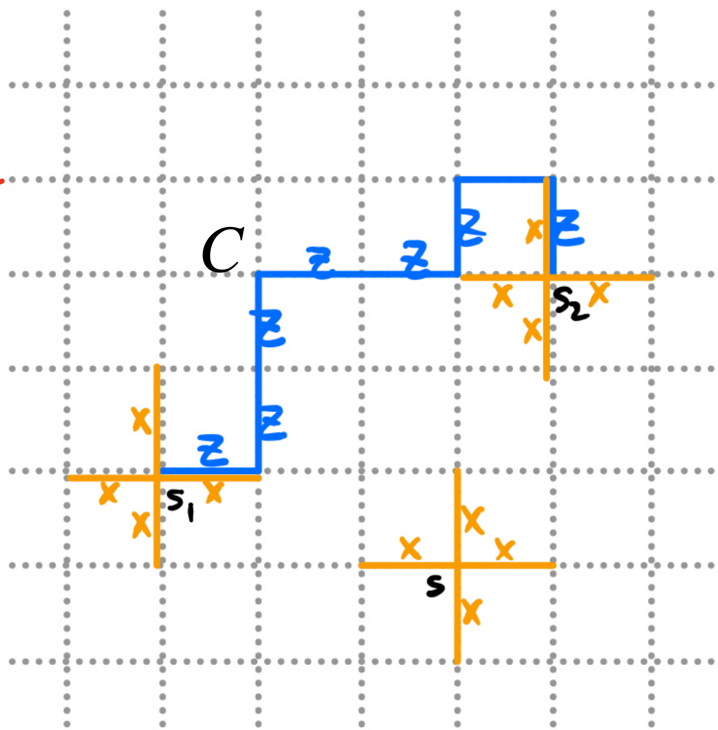
$$W_{C_x} V_{\tilde{C}_y} = - V_{\tilde{C}_y} W_{C_x}$$

e excitations

Endpoints of Wilson operator W_C are electric charge excitations e.

$A_s|GS\rangle = B_p|GS\rangle$
$= |GS\rangle$

$A_s = $ 



$$|e_{s_1} e_{s_2}\rangle = W_C|\mathrm{GS}\rangle$$

$A_{s_2} W_C|GS\rangle = -W_C A_{s_2}|GS\rangle$
$= -W_C|GS\rangle$

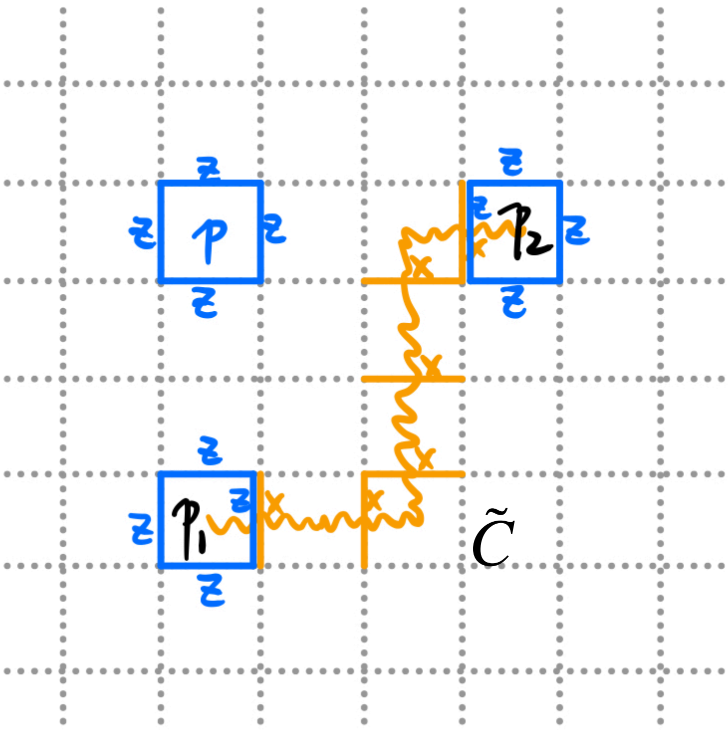
$$A_{s_1} = A_{s_2} = -1$$

$$A_{s \neq s_1, s_2} = 1$$

$$B_p = 1$$

m excitations

Endpoints of 't Hooft operator $V_{\tilde{C}}$ are magnetic charge excitations m.

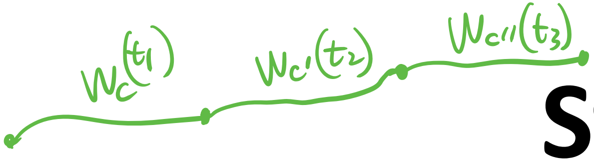


$$|m_{p_1}m_{p_2}\rangle = V_{\tilde{C}}|\mathrm{GS}\rangle$$

$$B_{p_1} = B_{p_2} = -1$$

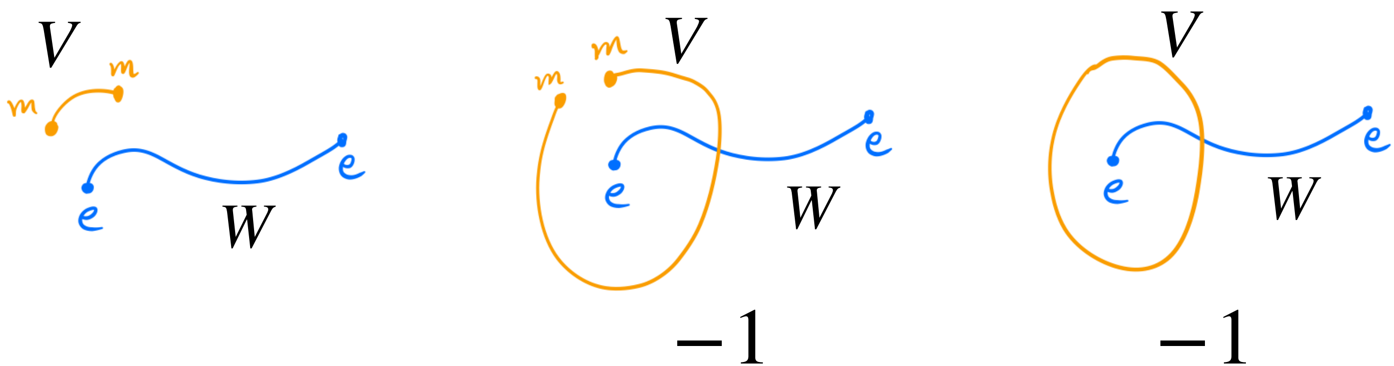
$$B_{p\neq p_1,p_2} = 1$$

$$A_s = 1$$



Statistics

- W_C and $V_{\tilde{C}}$ can be used to create / move e and m excitations
- Full braiding of e and m gives -1 sign



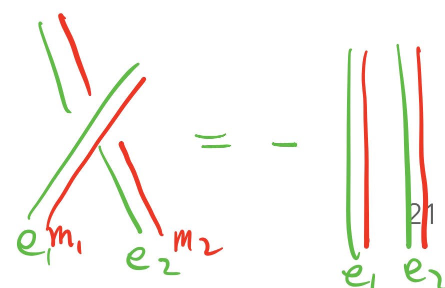
Anyons

- In (2+1) dimensions, point-like excitations (anyons) can have any rational number statistical phases.

$f = (e\, m)$ W_c $V_{\tilde{c}}$ ribbon operator (with framing)

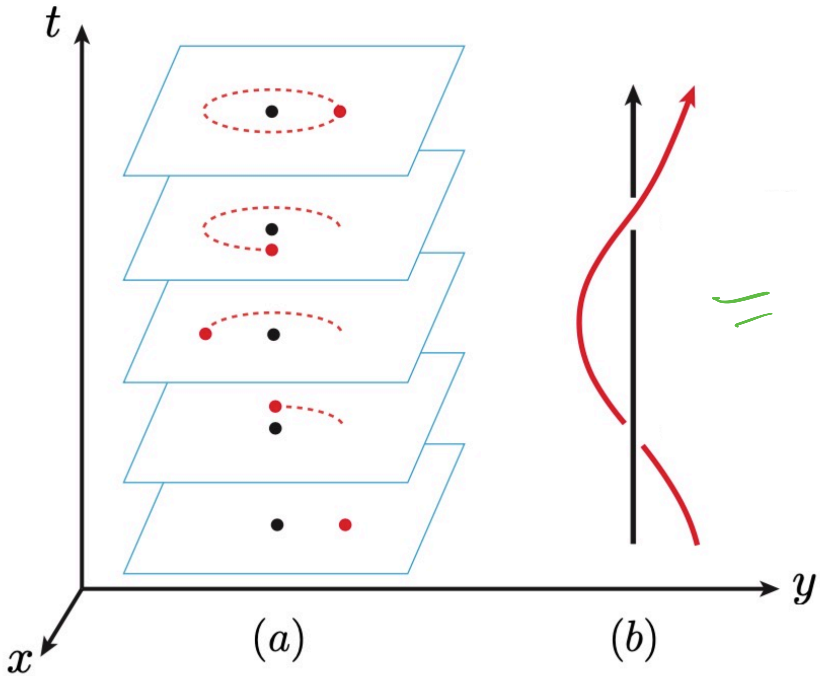
- Anyons in toric code: $1, e, m, f = em$
- Both e and m are bosons; f is fermion
- e, m, f have mutual braiding phase -1

boson/fermion: $e_1 \quad e_2 \Rightarrow \pm \; e_2 \quad e_1$



knot/link in (2+1)D spacetime

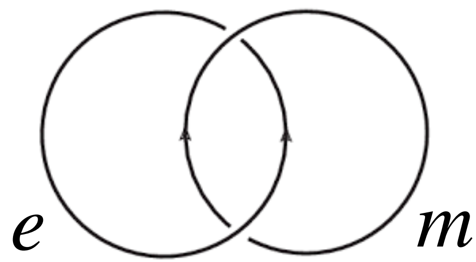
anyon worldlines in (2+1)D <—> knot/link in 3D



$= e^{i\theta_{anyon}}$

Hopf link

For the toric code model, mutual statistics of e and m:



$$= -1$$

In general, for a link of e and m worldlines, we have statistical phase $(-1)^{\text{linking \#}}$.

Anyon models -> knot/link invariants

Summary

- Toric code model is a commuting projector Hamiltonian on torus
- Ground state subspace $\cong H_1(T^2, \mathbb{Z}_2)$
- Anyons braiding statistics and knot/link invariants
- Generalizations of toric code model later…

Example 2 Haldane's honeycomb model

(Another Haldane chain : 1D spin-1 Heisenberg chain)

Nobel 2016

Haldane 1988: quantum Hall effect without Landau levels.

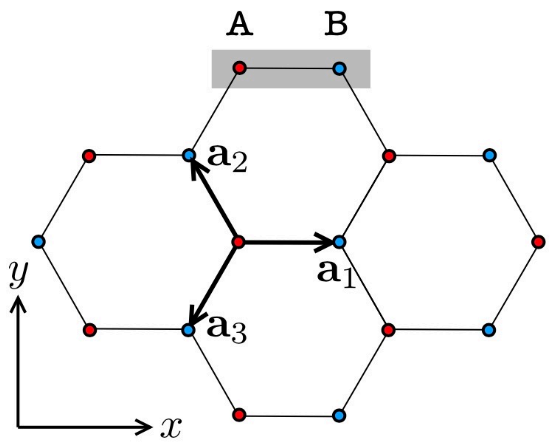
now also called: quantum anomalous Hall effect, Chern insulator.

Properties: (1) free fermion system, hopping on 2D honeycomb lattice

(2) break time reversal symmetry.

(3) has nontrivial topology (winding number / Chern number)

(4) nontrivial Hall conductance / nontrivial chiral edge state



spinless fermion hopping on the lattice.

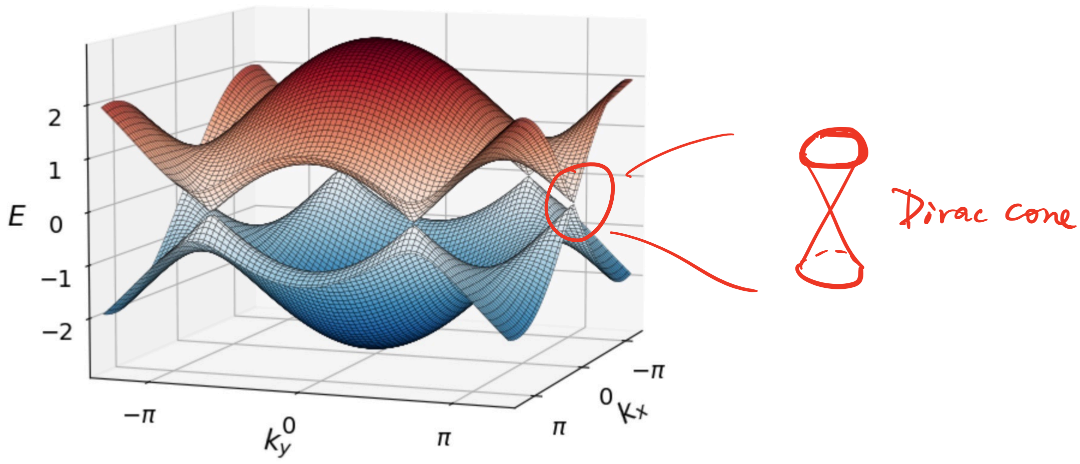
real space : $H_0 = \sum_{\langle ij \rangle} t\, c_i^\dagger c_j$

momentum space: $H_0 = \sum_{\vec{k}} H_0(\vec{k}) = \sum_{\vec{k}} \begin{pmatrix} 0 & h(\vec{k}) \\ h(\vec{k})^* & 0 \end{pmatrix}$

$h(\vec{k}) = t \sum_i e^{i\vec{k}\cdot\vec{a}_i}$

$= t \sum_i \cos(\vec{k}\cdot\vec{a}_i)\, \sigma_x - t \sum_i \sin(\vec{k}\cdot\vec{a}_i)\, \sigma_y$

Spectrum $E_0(\vec{k}) = \pm |h(\vec{k})|$

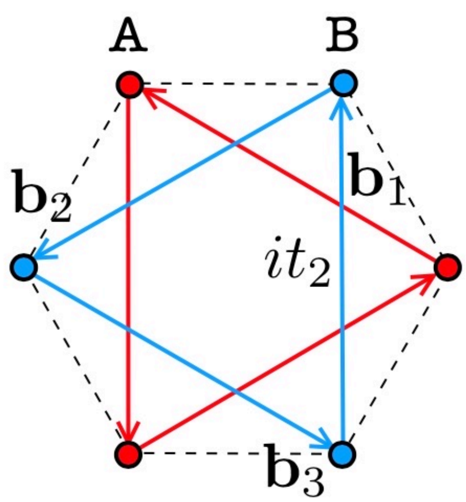


Add more terms to gap out H_0.

① opposite onsite energy $\pm M$ to sublattice A/B.

$$H(\vec{k}) = H_0(\vec{k}) + \underbrace{\begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}}_{= M\sigma_z}$$

② add second nearest neighbor hopping.



real : $H = t\sum_{\langle ij\rangle} c_i^\dagger c_j \pm it_2 \sum_{\langle\langle ij\rangle\rangle} c_i^\dagger c_j \pm M\sum_i c_i^\dagger c_i$

momentum : $H(\vec{k}) = H_0(\vec{k}) + M\sigma_z + 2t_2 \sum_i \sin(\vec{k}\cdot\vec{b}_i)\,\sigma_z$

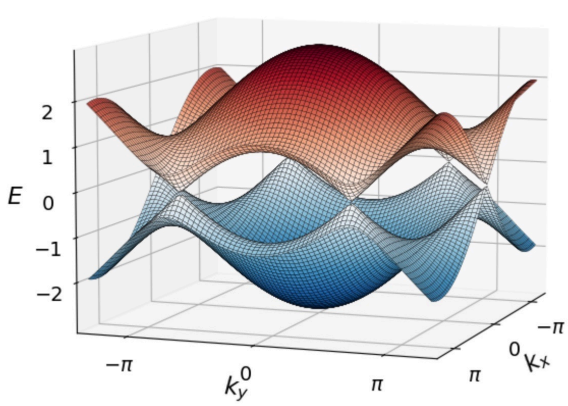
$$= \underbrace{t_1 \sum_i \cos(\vec{k}\cdot\vec{a}_i)}_{B_x(\vec{k})} \sigma_x \underbrace{- t_1 \sum_i \sin(\vec{k}\cdot\vec{a}_i)}_{B_y(\vec{k})} \sigma_y$$

$$+ \underbrace{\left[M + 2t_2 \sum_i \sin(\vec{k}\cdot\vec{b}_i)\right]}_{B_z(\vec{k})} \sigma_z$$

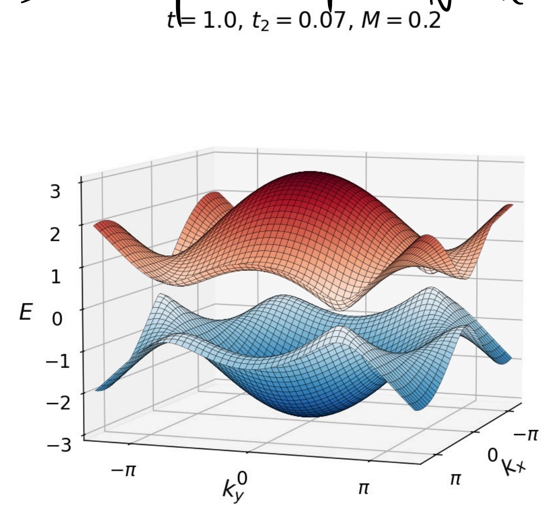
$B_z(\vec k)$

$$= \vec{B}(\vec{k}) \cdot \vec{\sigma}$$

$\Rightarrow$ spectrum $E(\vec{k}) = \pm \left| \vec{B}(\vec{k}) \right| = \pm \sqrt{B_x(\vec{k})^2 + \cdots + \cdots}$



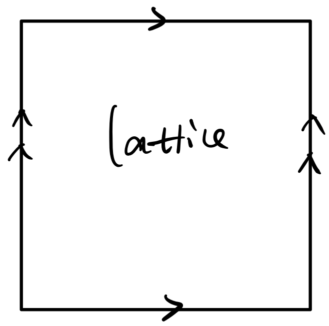
$t_2 = 0$



$t_2 \neq 0$

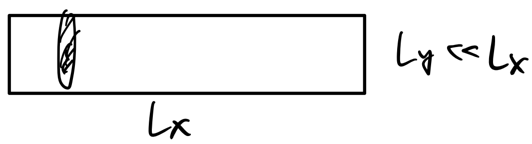
$\vec{k} \in BZ$

$\vec{k} = 0 \sim \vec{k} = 2\pi$

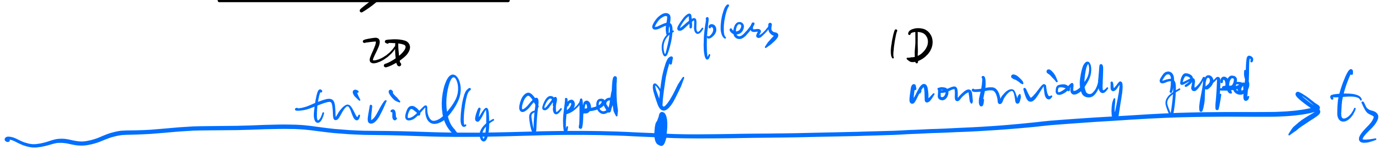


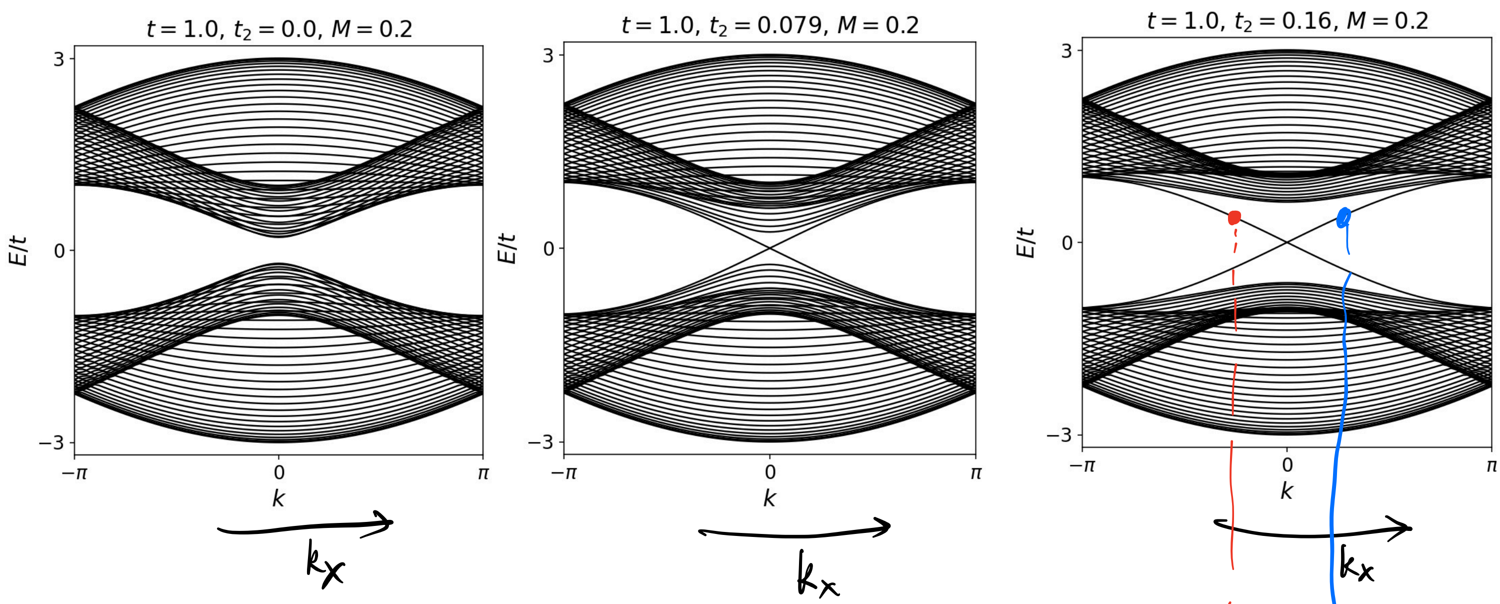
2D

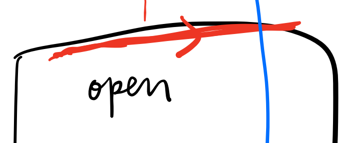
$\Rightarrow$

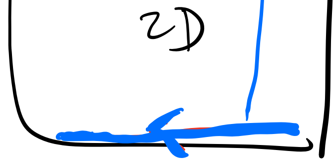


1D



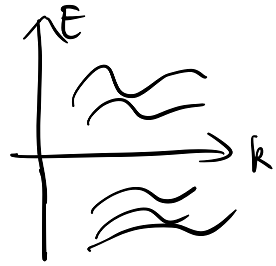






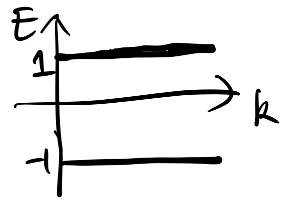
$$H(\vec{k}) = \vec{B}(\vec{k}) \cdot \vec{\sigma}$$

$$E(\vec{k}) = \pm|\vec{B}(\vec{k})|$$



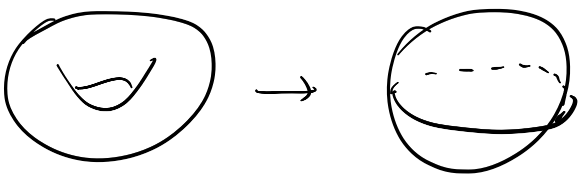
nomalize $\hat{H}(\vec{k}) = \dfrac{\vec{B}(\vec{k}) \cdot \vec{\sigma}}{|\vec{B}(\vec{k})|}$

$$\hat{E}(\vec{k}) = \pm 1$$



$$\hat{H} : T^2 = BZ \longrightarrow S^2$$

$$\vec{k} \longmapsto \hat{H}(\vec{k})$$



Homotopy type of the map $\hat{H}$ determines the topologial properties of the Haldane model.

Berry connection $\Rightarrow$ winding number / Chern number.